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A combinatorial approach to the Perrin polynomial sequence and its generalization

Uma abordagem combinatória para a sequência polinomial de Perrin e sua generalização

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Abstract

The present work approaches the study of a combinatorial interpretation for the Perrin polynomial sequence, aiming at a contribution to the Mathematics teaching area. In view of studies focused on the History of Mathematics containing the teaching of recurring numerical sequences present in textbooks, there is an interest in carrying out a combinatorial approach for a sequence considered to be a cousin of the Fibonacci sequence, which is the Perrin sequence. In this way, we have the definition of Perrin's polynomial bracelets, investigating Perrin's polynomial combinatorial model. Finally, combinatorial interpretations for the Perrin polynomial sequence extensions are discussed, generalizing the Perrin polynomial combinatorial model.

Keywords: combinatorics; mathematics teaching; Perrin polynomial sequence.

Resumo

O presente trabalho aborda o estudo de uma interpretação combinatória para a sequência polinomial de Perrin, visando uma contribuição para a área de ensino de Matemática. Tendo em vista estudos voltados para a História da Matemática contendo o ensino de sequências numéricas recorrentes presentes em livros didáticos, há interesse em realizar uma abordagem combinatória para uma sequência considerada prima da sequência de Fibonacci, que é a sequência de Perrin. Dessa forma, tem-se a definição das pulseiras polinomiais de Perrin, investigando o modelo combinatório polinomial de Perrin. Por fim, são discutidas interpretações combinatórias para as extensões da sequência polinomial de Perrin, generalizando o modelo combinatório polinomial de Perrin.

Palavras-chave: combinatória; ensino de matemática; sequência polinomial de Perrin.

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1 Introduction

This paper discusses and defines a combinatorial interpretation related to the Perrin polynomial sequence. Thus, the reader will follow discussions on the Tetarrin polynomial sequences and their generalizations, obtained by adding terms to their recurrence. This extends the classical Perrin numbers. It should also be noted that the combinatorial approach originates from the notion of the board, defined by Spreafico [1].

Introducing a brief study around the sequence that will be addressed, we have that the Perrin polynomial sequence is a third order sequence, is a third–order numerical recurrence given by: $R_n(x) = xR_{n-2}(x) + R_{n-3}(x), R_0(x) = 3, R_1(x) = 0, R_2(x) = 2, n \geq 3$. By assigning the value of $x = 1$, we have the original Perrin sequence, created by Raoul Perrin [2].

Thus the first terms presented in Table 1.

Table 1: Perrin's first ten polynomial terms.

n	$R_n(x)$
0	3
1	0
2	2
3	3
4	$2x$
5	$3x + 2$
6	$2x^2 + 3$
7	$3x^2 + 4x$
8	$2x^3 + 6x + 2$
9	$3x^3 + 6x^2 + 3$

Source: Elaborated by the authors.

In view of this, it is important to remember the sequence of Padovan (P_n). The Padovan sequence has the same recurrence formula as the Perrin sequence, differentiating their initial values $(P_0 = P_1 = P_2 = 1)$. These two sequences also present other similarities, which can be deepened in the work of Vieira [3].

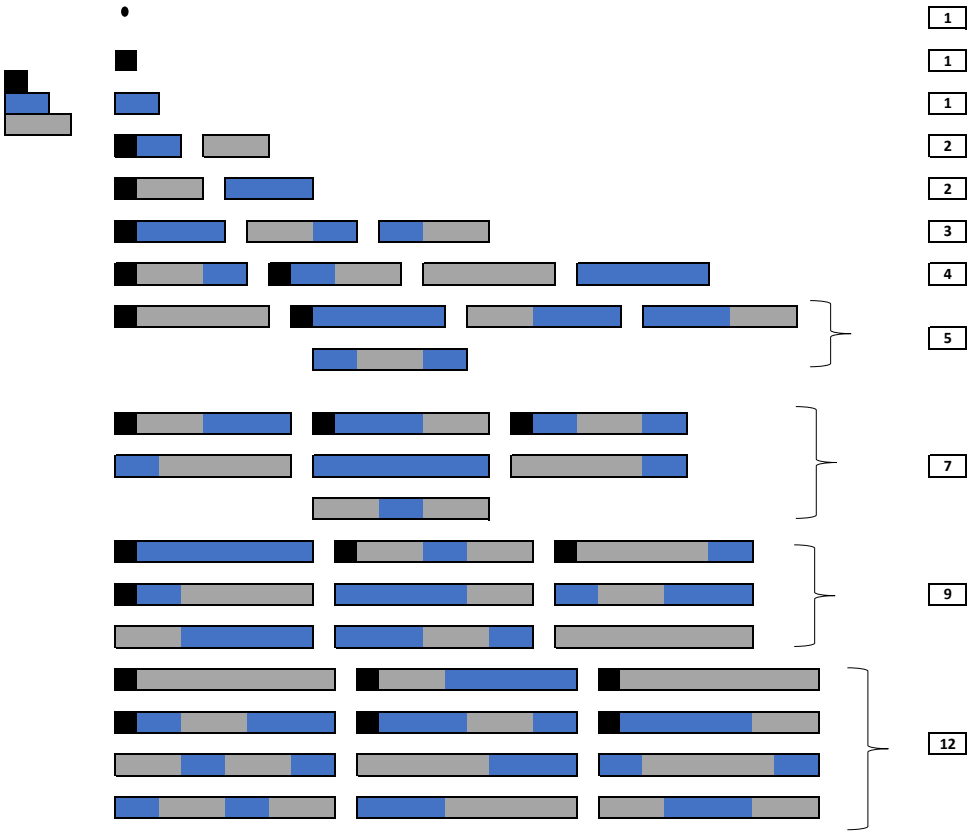
Vieira, Alves and Catarino [4] in their research, they defined the Padovan combinatorial model, with the Padovan board, based on a defined by the following construction rule: Considering an extended domino, defined as a rectangle of size 1×3; a 1×1 black square and the 1×2 domino, any Padovan tile is formed by arranging the defined shapes. The black square is intended to complement empty tiles, subject to the rule that it is inserted only at the beginning and only once on each tile. The particular rules mentioned are defined for the theorem concerning Padovan tiling [4].

Thus a n-board is considered, with the following tile shapes: black square 1×1, blue dominoes 1×2, gray extended dominoes 1×3, all with weight 1. In Figure 1, on the left side, some examples are provided in order to fill in the n-board corresponding to the Padovan sequence. On the right side are the terms corresponding to the Padovan numbers. With this it is possible to perceive the term p_n as being the amount of tile shapes on the n-board, following the aforementioned rules, determining the relation: $p_n = P_n, n \geq 0$. Next, $p_0 = 1 = P_0$ is fixed, when there is no board of size $1 \times n$, and $p_1 = 1 = P_1$, when there is only one black square. Thus there is an exception to the rule, since in this case, the black square will be the only figure, being at the beginning and consequently at the end (simultaneously).

In fact, there are the following propositions related to Padovan tiles, being therefore: the number of tiles that start with black squares corresponds to a number of the Padovan sequence, being, given by the formula p_{n-3}. However, the number of tiles that do not start with a black square is given by p_{n-2} (see Figure 1).

For $n \geq 0$, the possible tiling of a $1 \times n$ board, starting with the black square tiles is given by: p_{n-3}. And for $n \geq 0$, the possible tilings of a $1 \times n$ board, which do not start with the black square tiles is given by: p_{n-2}.

Figure 1: Padovan tiling.



Source: [4].

In this way, polynomial Perrin bracelets are primarily defined in this research, presenting Perrin's polynomial combinatorial model as a combinatorial interpretation for the recurrent numerical sequence and its generalization. The polynomial formulation proposed here is novel and is introduced for the first time in this article.

2 Perrin's polynomial combinatorial model

In a peculiar way to Padovan's polynomial combinatorial model and its generalization with the Z-dovan sequence, there is research around the Perrin polynomial sequence. Thus we have the contributions of the present investigative study, referring to Perrin's polynomial combinatorial model, its extensions and generalization.

With the discussions carried out in the introduction of this research and in its highlighted sources, we will approach Padovan's combinatorial model theorem, since the sequences have similarities.

Theorem 1. *For $n \geq 0$, the possible tiling of a $1 \times n$ board, with black square, blue domino and gray extended domino tiles is given by:*

$$p_n = P_n,$$

where p_n is the number of ways to fill the board $1 \times n$ and P_n is the nth term of the Padovan sequence [4].

Proof. Based on the arguments and demonstrations of [4], we have that:

Let p_n be the sum of the tiling screens, where $p_0 = 1 = P_0$, $p_1 = 1 = P_1$.

Considering the tiles of size $n-2$, where $n \geq 0$, we have:

Case 1: Let be the subsets of the tilings that start with the black square and another subset of the other tilings. Consider a new set of tiles, starting with the blue domino, of size 1×2, juxtaposed with the tiles of size $n-2$ that do not start with the black square in union with the set of tiles that start with the black square , of size 1×1, inserting a blue domino of size 1×2, right after this square. This union results in a tiling set of size n (Figure 2).

Figure 2: Union of trays of size $n-2$.



Source: [4].

Thus the number of tiles starting with the blue domino is given by p_{n-4}, while the number of tiles starting with the black square, followed by the blue domino, is given by p_{n-5} . The join is given by p_{n-2}.

Case 2: Consider the subsets of tiling that start with the extended dominoes (tiles that do not start with the black squares) and another subset of the tilings that start with the black square, followed by the extended domino. Consider a new set of tiles, starting with the extended gray domino, of size 1×3, juxtaposed with the tiles of size $n-3$ that do not start with the black square in union with the set of tiles that start with the square black, size 1×1, inserting a gray extended domino of size 1×3, right after this square. This union results in a tiling set of size $n-3$ (Figure 3).

Figure 3: Union of trays of size $n-3$.



Source: [4].

Thus the number of tiles starting with the gray extended domino is given by p_{n-5}, while the number of tiles starting with the black square followed by the gray extended domino is given by p_{n-6}. The join is given by p_{n-3}.

Therefore, by the principle of adding the analyzed independent cases (without using the intersection), $p_n = p_{n-2} + p_{n-3}$. Satisfying the Padovan recurrence ($P_n = P_{n-2} - P_{n-3}$), with the initial conditions $p_0 = P_0$ and $p_1 = P_1$. Soon $p_n = P_n$. □

In order to carry out a study regarding Perrin's polynomial combinatorial model, the relationship between the polynomial sequence of Padovan and Perrin has been obtained, from the resolution by linear systems. Thus we have the algebraic relationship:

$$R_{n+2}(x) = \frac{2}{x}P_n(x) + \frac{2x-3}{x}P_{n+1}(x) + \frac{1}{x}P_{n+2}(x) \tag{1}$$

For the study of Perrin's polynomial combinatorial model, Newton's binomial is used, defined as a form of Padovan's polynomial combinatorial model.

Definition 2. *Padovan's polynomial combinatorial model is given by:*

$$P_n(x) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{O(n,k)}{k} x^k,$$

where $O(n,k) = \lfloor \frac{n+(k-1)}{3} \rfloor$.

Definition 3. *For $x \in \mathbb{Z}$, the Padovan polynomial is defined, with $n \geq 3, n \in \mathbb{N}$, by recurrence:*

$$P_n(x) = xP_{n-2}(x) + P_{n-3}(x),$$

where $P_0(x) = P_1(x) = P_2(x) = 1$.

Thus the first terms presented in Table 2.

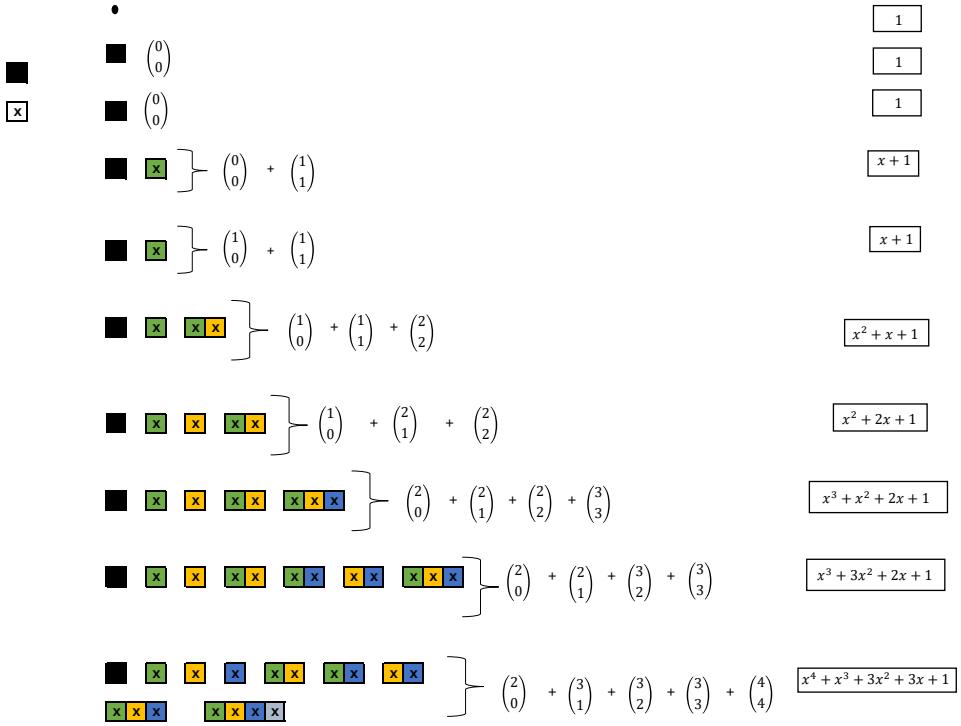
Table 2: Padovan's first ten polynomial terms.

n	$P_n(x)$
0	1
1	1
2	1
3	$x+1$
4	$x+1$
5	x^2+x+1
6	x^2+2x+1
7	x^3+x^2+2x+1
8	x^3+3x^2+2x+1
9	$x^4+x^3+3x^2+3x+1$

Source: Prepared by the authors.

From Definition 3 and 2, a combinatorial interpretation is established, considering the colored tiles of Padovan and assigning weight x to the colored square. In the Figure 4, on the left side, examples are presented with the bias of filling a corresponding n-board with the Padovan polynomial sequence and its binomial (see Definition 2).On the right side, that the corresponding expressions can be compared with the elements indicated in the Table 2 and then the Theorem 4 provides a combinatorial basis for counting the number of shapes to fill, via tiles, corresponding to the number of tile shapes for each board an algebraic expression corresponding to the weight.

Figure 4: Polynomial Padovan tiling.



Source: Prepared by the authors.

Based on the Definition 2, a new definition has been formed referring to Padovan's colored tiling.

Definition 4. *Considering a board of the type* $1 \times n$ *and the black square* 1×1*, with weight* 1*, while the colored square* 1×1*, with weight x, as shapes of tiles. Then, the number of ways to fill this board, matching their respective weights, will be determined by* $p_n(x) = P_n(x)$*, for* $n \geq 0$*.*

In this way, it was possible to define and discuss the first terms of Padovan's polynomial numbers, based on the generalization of the mathematical combinatorial formula presented, allowing an investigative study of Padovan's combinatorial model.

Definition 5. *Perrin's polynomial combinatorial model is given by:*

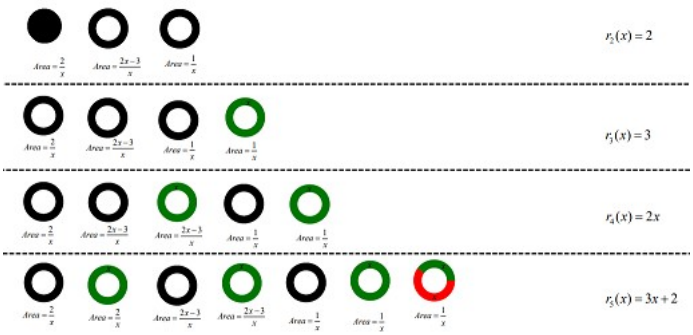
$$R_{n+2}(x) = \frac{2}{x}\sum_{k=0}^{\lceil \frac{n}{2} \rceil}\binom{O(n,k)}{k}x^k + \frac{2x-3}{x}\sum_{k=0}^{\lceil \frac{n+1}{2} \rceil}\binom{O(n+1,k)}{k}x^k + \frac{1}{x}\sum_{k=0}^{\lceil \frac{n+2}{2} \rceil}\binom{O(n+2,k)}{k}x^k,$$

where $O(n,k) = \lfloor \frac{n+(k-1)}{3} \rfloor$.

From the Definition 5 and the colored tiling referring to Padovan's polynomial model, it is possible to establish the study of bracelets, for Perrin's polynomial combinatorial model. The bracelets are formed by integrating the area (given by the coefficients of the Equation 1), along with the combinations of Padovan's polynomial model.

Revisiting Padovan's polynomial combinatorial interpretation, we consider the black square of weight 1 and assign weight x to the colored square. Padovan's bracelet is fitted immediately after a circle of area defined by the coefficients in Equation 1. The height of Padovan's bracelet is irrelevant, with the proportional adjustment of the tiles being the fundamental characteristic for this representation. To exemplify Perrin's polynomial bracelet, see Figure 5.

Figure 5: Perrin polynomial model bracelet.



Source: Prepared by the authors.

Theorem 6. *For* $n \geqslant 0$*, the possible bracelets of a circular Perrin polynomial board, of size* $1 \times n$*, with the area established by the coefficients of Equation 1 and the black square* 1×1*, with weight* 1*, while the colored square 1 x 1, with weight x is given by:*

$$r_n(x) = R_n(x),$$

where $r_n(x)$ is the number of ways to fill the bracelet $1 \times n$ from the Perrin polynomial and $R_n(x)$ is the nth term of the Perrin polynomial sequence.

Proof. In a similar way to Theorem 1, we have the validation of this theorem. □

3 Tetrarrin's polynomial combinatorial model

In this section, the extension of the polynomial Padovan sequence is examined, which is called the polynomial Tridovan, based on the definitions established by [5, 6, 7].

Definition 7. *The polynomial Tetrarrin sequence, $R_{3(n)}(x)$, satisfies the following recurrence formula, for $n \in \mathbb{N}$ and $n \geqslant 4$.*

$$R_{4(n)}(x) = x^2 R_{4(n-2)}(x) + xR_{4(n-3)}(x) + R_{4(n-4)}(x),$$

with the initial terms: $R_{4(0)}(x) = 3, R_{4(1)}(x) = 0, R_{4(2)}(x) = 2x^2, R_{4(3)}(x) = 3x$.

Thus we have the Table 4 with the first terms of the polynomial Tetrarrin sequence.

Table 3: First ten polynomial terms of Tetrarrin.

n	$R_{4(n)}(x)$
0	3
1	0
2	$2x^2$
3	$3x$
4	$2x^4 + 3$
5	$5x^3$
6	$2x^5 + 8x^2$
7	$7x^5 + 6x$
8	$2x^7 + 15x^4 + 3$
9	$7x^7 + 2x^6 + 13x^3 + 6x^2$

Source: Prepared by the authors.

With this the relationship between the polynomial sequences of Tridovan and Tetrarrin is investigated, through the resolution of linear systems, obtaining:

$$R_{4(n+2)}(x) = -x^2 P_{3(n)}(x) + 3P_{3(n+2)}(x) \tag{2}$$

Therefore, the study of Tetrarrin's polynomial combinatorial model can be established, based on the Theorem 9, which depicts Tridovan's polynomial combinatorial model.

Definition 8. *The polynomial Tridovan sequence, $P_{3(n)}(x)$, satisfies the following recurrence formula, for $n \in \mathbb{N}$ and $n \geq 4$.*

$$P_{3(n)}(x) = x^2 P_{3(n-2)}(x) + xP_{3(n-3)}(x) + P_{3(n-4)}(x),$$

with the initial terms: $P_{3(0)}(x) = 1, P_{3(1)}(x) = 0, P_{3(2)}(x) = x^2, P_{3(3)}(x) = x$.

Thus we have the Table 4 with the first terms of the polynomial Tridovan sequence.

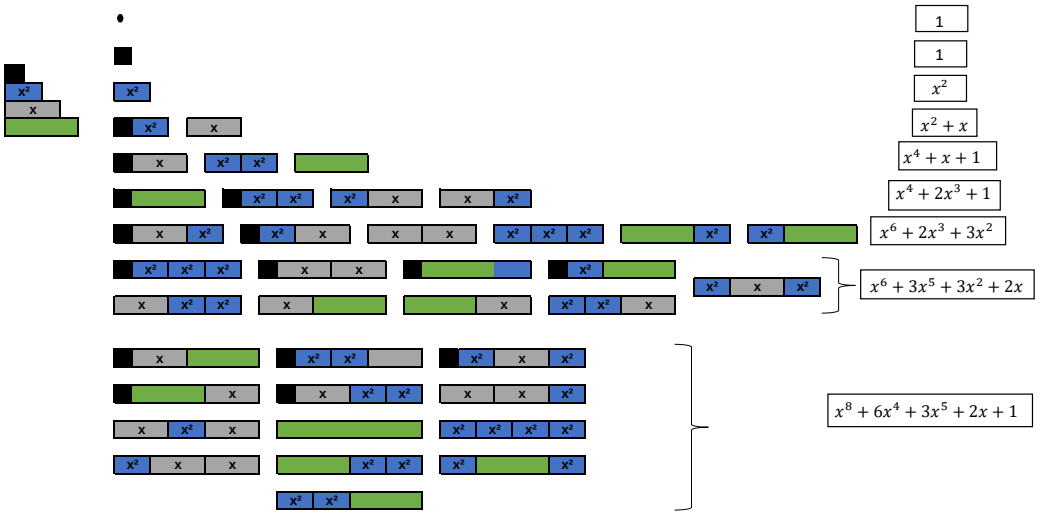
Table 4: Tridovan's first ten polynomial terms.

n	$P_{3(n)}(x)$
0	1
1	0
2	x^2
3	x
4	$x^4 + 1$
5	$2x^3$
6	$x^6 + 3x^2$
7	$3x^5 + 2x$
8	$x^8 + 6x^4 + 1$
9	$4x^7 + 7x^3$

Source: Prepared by the authors.

To establish its combinatorial interpretation, consider the black 1×1 square with weight 1, the blue 1×2 domino with weight x^2, the gray extended domino 1×3 with weight x and the double green domino 1×4 with weight 1. Fixing the referred pieces in a preliminary way, in Figure 6 the reader will be able to visualize some configurations and initial boards that, on the right side, there is the correspondence of its final weight to the polynomials from Tridovan.

Figure 6: Polynomial Tridovan tiling.



Source: Prepared by the authors.

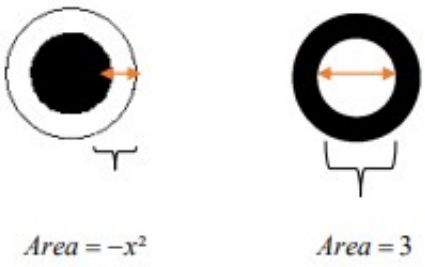
Theorem 9. *Considering a board of the type* $1 \times n$ *and the black square* 1×1*, with weight* 1*, while the blue domino* 1×2*, with weight* x^2*, the gray extended domino* 1×3 *with weight* x *and*

the double green 1×4 *domino with weight* 1*, as tile shapes for the Tridovan polynomial sequence. Then, the number of ways to fill this board, matching their respective weights, will be determined by* $p_{3(n)}(x) = P_{3(n)}(x) + P_{3(n-1)}(x)$*, for* $n \geq 0$*.*

Proof. In a similar way to Theorem 1, we have the validation of this theorem. □

The polynomial combinatorial model of Tetrarrin is given by integrating the combinatorial polynomial model of Tridovan, together with the bracelets already discussed for the combinatorial polynomial model of Perrin. Since the presented relation of the Equation 2 has coefficients belonging to integer numbers, it is clear that the bracelets made available for the circular Tetrarrin board will be arranged in two ways, namely: for positive coefficients, the Tridovan's bracelet is set immediately after a circle of area corresponding to the coefficient value; for negative coefficients, the Tridovan bracelet is arranged in the form of circular sectors with a joint area equal to one unit, adjusted immediately after the sectors, there is a bracelet with an area corresponding to the module of the corresponding coefficient. Thus for positive coefficients, the bracelet is given by the concentric circle with an area equal to the coefficient. For negative coefficients, there is a graph in the form of a pie, divided according to the same rule existing in the bracelet, with a concentric ring of area equal to the module of the coefficient, see Figure 7.

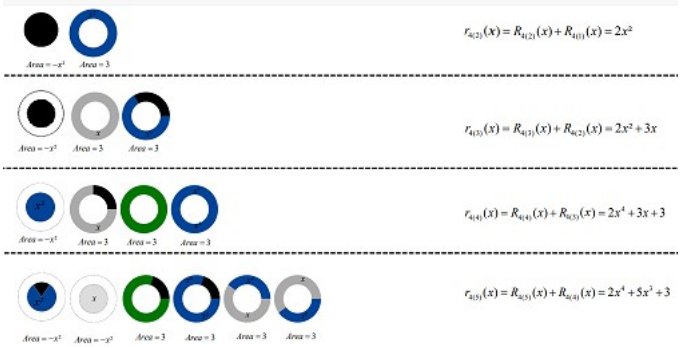
Figure 7: Example of the areas of Tetrarrin's polynomial bracelets.



Source: Prepared by the authors.

In general, we have the polynomial combinatorial interpretation of Tridovan, in which it presents its respective configuration given by: black square 1×1 with weight 1, blue domino 1×2 with weight x^2, gray extended domino 1×3 with weight x and double green dominoes 1×4 with weight 1. Fixing the aforementioned pieces in a preliminary way, in Figure 8 it is possible to visualize some configurations and initial bracelets that, on the right side, have the correspondence of its final weight to the Tridovan polynomials.

Figure 8: Tetrarrin Polynomial Combinatorial Model Bracelet.



Source: Prepared by the authors.

Theorem 10. *For $n \geq 0$, we have that the possible bracelets of a circular Tetrarrin polynomial board, of size $1 \times n$, with the area established by the coefficients of Equation 2 and the black square 1×1 with weight 1, the blue 1×2 domino with weight x^2, the extended gray 1×3 domino with weight x, and the green 1×4 double domino with weight 1, given by:*

$$r_{4(n)}(x) = R_{4(n)}(x) + R_{4(n-1)}(x),$$

where $r_{4(n)}(x)$ is the number of shapes to fill the bracelet $1 \times n$ of the Tetrarrin polynomial and $R_{4(n)}(x)$ is the nth term of the Tetrarin polynomial sequence.

Proof. In a similar way to Theorem 1, we have the validation of this theorem. □

It is noteworthy that the size of the figures presented are the same size in order to preserve the area to be calculated.

Next, the study of the generalized form of the combinatorial model of Perrin's polynomial numbers is carried out.

4 The Z-th-rrin polynomial combinatorial model

In order to obtain the generalized form of the Perrin polynomial model, the Z-thirteenth polynomial combinatorial model ($R_{z(n)}$) is defined. Since the combinatorial interpretation of Perrin's polynomial sequence is based on Padovan's color tiling, its study is individualized. Thus from the previously studied Tetrarrin polynomial combinatorial model, the generalization of the combinatorial form of the Z-th-rrin polynomial sequence is studied.

Based on the conjecture which portrays the existence of a linear combination between the polynomial sequences of Z-dovan ($P_{z(n)}$) and Z-th-rrin of the same order, the system of equations similarly to the Z-th-rrin combinatorial model.

Through this, it is possible to establish the Z-th-rrin polynomial combinatorial interpretation, which is defined as a combination of the bracelets of the Z-dovan polynomials, where the number of units of a given bracelet is determined by the coefficient associated with the polynomial term

of Z-dovan to which he belongs. The multiplicative property of the coefficients is represented by calculating the area and, in this way, one can have coefficients belonging to the set of integers. Given this, it appears that, for positive coefficients, Z-dovan's bracelet is adjusted immediately after a circle of area corresponding to the value of its coefficient; for negative coefficients, Z-dovan bracelet is arranged in the form of circular sectors of joint area of unitary value and, adjusted immediately after the sectors, we have a bracelet whose area corresponds to the module of the respective coefficient. So, for positive coefficients, the bracelet formed by a concentric circle with an area equal to the coefficient is presented. For negative coefficients, there is a graph in the form of a pie, divided according to the same rule existing in the bracelet, with a concentric ring of area equal to the module of the coefficient.

Then, there is the corollary then, determining the generalization of the Perrin polynomial sequence, by means of the Z-essimorrin polynomial sequence. It should be noted that the generalization of Z-th-rrin polynomial tilings is given from the Tetrarrinn polynomial sequence.

Corollary 11. *For $n \geq 5$, the possible bracelets of a circular board of size $1 \times n$, with z coins of different colors and the black square tiles 1×1, blue dominoes 1×2, 1×3 gray extended dominoes, 1×4 green double extended dominoes, 1×5 yellow extended dominoes ... and an orange $1 \times z$ square is given by:*

$$r_{z(n)}(x) = R_{z(n)}(x) + R_{z(n-1)}(x),$$

where z_n is the number of shapes to fill the board $1 \times n$ and $R_{z(n)}(x)$ is the nth term of the Z-th-rrin polynomial sequence.

5 Conclusion

The present research enabled a study of Perrin's combinatorial interpretation, introducing Perrin's polynomial combinatorial model of numbers. In this way, it is possible to explore the combinatorial approach of the recurrent numerical sequence, contributing to the teaching of Mathematics in the History of Mathematics.

From the Perrin polynomial sequence, it was possible to define its respective combinatorial model and, by inserting elements in the main recurrence, define the Tetrarrin polynomial sequence and its generalization. After that, the combinatorial models of the other sequences derived from Perrin were introduced until their Z-th-rrin polynomial generalization.

For future research, we seek to develop didactic teaching situations, allowing application in the classroom in initial training courses for Mathematics teachers.

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