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Analytic conjugation between planar differential systems and potential systems*

Conjugação analítica entre sistemas diferenciais planares e sistemas potenciais**

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Abstract

The classic Poincaré Normal Form Theorem states that a critical point of an analytic planar vector field is a non-degenerate center if and only if there is an analytic coordinate change such that in the new coordinates the vector field initial is of the form $f(x^2+y^2)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$, where f is an analytic function defined in a neighborhood of the origin such that $f(0) > 0$. In this article it is proved that an analytical planar vector field with a non-degenerate center at $(0,0)$ is analytically conjugate, in a neighborhood of $(0,0)$, to a Hamiltonian vector field of the form $y\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial y}$, where V is an analytic function defined in a neighborhood of the origin such that $V(0) = V'(0) = 0$ and $V''(0) > 0$. This result is a partial answer to a question proposed by Chicone in 1987.

Keywords: Analytic planar vector fields. Non-degenerate center. Analytic conjugation. Potential systems.

Resumo

O clássico Teorema da Forma Normal de Poincaré afirma que um ponto crítico de um campo vetorial planar analítico é um centro não degenerado se e somente se houver uma mudança de coordenada analítica tal que nas novas coordenadas o campo vetorial inicial seja da forma $f(x^2+y^2)\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$, onde f é uma função analítica definida em uma vizinhança da origem tal que $f(0) > 0$. Neste artigo é provado que um campo vetorial planar analítico com um centro não degenerado em $(0,0)$ é analiticamente conjugado, em uma vizinhança de $(0,0)$, a um campo vetorial hamiltoniano da forma $y\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial y}$, onde V é uma função analítica definida em uma vizinhança da origem tal que $V(0) = V'(0) = 0$ e $V''(0) > 0$. Este resultado é uma resposta parcial a um problema proposto por Chicone em 1987.

Palavras-chave: Campos vetoriais planares analíticos. Centro não degenerado. Conjugação analítica. Sistemas potenciais.

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1 Introduction

Let Ω be an open subset of $\mathbb{R}^2$ and $C^\omega(\Omega, \mathbb{R}^d)$ the set of real analytic functions defined on Ω with values in $\mathbb{R}^d$, $d \in \{1, 2\}$. Let $P, Q \in C^\omega(\Omega, \mathbb{R})$ and consider the analytic differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (x, y) \in \Omega. \tag{1}$$

The system (1) defines in Ω the planar vector field $X = (P, Q) \in C^\omega(\Omega, \mathbb{R}^2)$. In this article, the vector field $X = (P, Q)$ will often be represented by the differential operator

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}. \tag{2}$$

A point $p \in \Omega$ such that $X(p) = (0, 0)$ is called a singular point of X. A singular point p is non-degenerate if the determinant of the Jacobian matrix $DX(p)$ of X at p is nonzero, that is, if $P_x(p)Q_y(p) - P_y(p)Q_x(p) \neq 0$. A singular point p is called a center of X if there exists an open neighborhood U of p such that every solution of (1) with an initial condition in $U - \{p\}$ defines a periodic orbit around p. The largest neighborhood $\mathcal{A}$ with this property is called the period annulus of p. Let p be a center of X, and let $T(q)$ be the period of the orbit passing through $q \in \mathcal{A}$. The function $q \longrightarrow T(q)$ is called the period function associated with the center p.

Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^2$. The vector fields $X \in C^\omega(\Omega_1, \mathbb{R}^2)$ and $Y \in C^\omega(\Omega_2, \mathbb{R}^2)$ are analytically equivalent (or analytically conjugate) if there exists an analytic diffeomorphism $h : \Omega_1 \longrightarrow \Omega_2$ such that

$$D_q hX(q) = Y(h(q)) \quad \text{for every} \quad q \in \Omega_1. \tag{3}$$

The diffeomorphism h maps singular points to singular points and periodic orbits to periodic orbits, preserving the period of the periodic orbits. Let p be a singular point of X. We say that X is locally analytically conjugate to a vector field Y if equality (3) holds in a neighborhood of p.

2 Main Result

The main result of this paper is the following theorem.

Theorem 2.1 *Let Ω be an open subset of $\mathbb{R}^2$ such that $(0, 0) \in \Omega$, and suppose that the vector field $X \in C^\omega(\Omega, \mathbb{R}^2)$ has a non-degenerate center at $(0, 0)$. Then X is analytically conjugate, near $(0, 0)$, to the vector field*

$$Y = y\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial y}, \tag{4}$$

where V is an analytic function defined near the origin such that $V(0) = V'(0) = 0$ and $V''(0) > 0$.

Theorem 2.1 provides a partial answer to a problem proposed by Chicone in 1987 (see [1]).

The vector field (4) defines the Hamiltonian system

$$\dot{x} = y, \quad \dot{y} = -V'(x). \tag{5}$$

This system is called a potential system and has been the subject of study by many researchers. Among the issues addressed, two stand out, both related to the period function associated with the center of system (5). The first issue concerns the study of the monotonicity of the period function. This question has been addressed in several papers, especially [1], [2], [3], [4], [5]. The second issue pertains to the possibility of constructing a potential function V from a positive function T. This question is known as the inverse problem and has been addressed in several papers, especially [6], [7], [8], [9], [10], [11], [12]. In [1] Chicone asks under what conditions a vector field X with a center at $(0,0)$ is conjugate to a Hamiltonian vector field of type (4). Theorem (2.1) provides an answer to Chicone's question for the particular case where X is analytic and the center at $(0,0)$ is non-degenerate.

3 Proof of Theorem 2.1

The central idea of the proof is to construct a vector field of the type (4), starting from the period function of the vector field X. This construction ensures that the period function of Y is equal to the period function of X. The proof is divided into several lemmas.

Lemma 3.1 *Let $X = P\frac{\partial}{\partial u} + Q\frac{\partial}{\partial v}$ be an analytic vector field with a non-degenerate center at $(0,0)$. Then X is analytically conjugate, near $(0,0)$, to the vector field*

$$X(\xi,\eta) = f(\xi^2+\eta^2)\left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right), \tag{6}$$

where f is an analytic function defined near the origin such that $f(0) > 0$.

Proof: This is an immediate consequence of the Poincaré Normal Form Theorem (see [13]).

Note that (6) is a Hamiltonian vector field with the Hamiltonian function $H(\xi,\eta) = F(\xi^2+\eta^2)$, where F is the analytic function defined by

$$F(z) = \frac{1}{2}\int_0^z f(s)ds. \tag{7}$$

Therefore, the periodic orbits of (6) are contained in the level curves $H(\xi,\eta) = E$.

Lemma 3.2 *Let $E \geq 0$, the period function $T(E)$ of (6) parametrized by $H(\xi,\eta) = E$ is the analytic function given by*

$$T(E) = \frac{\pi}{F'(F^{-1}(E))} = \pi\frac{d}{dE}F^{-1}(E). \tag{8}$$

Proof: In polar coordinates, (6) becomes

$$X(r,\theta) = f(r^2)\frac{\partial}{\partial\theta}, \tag{9}$$

where $r^2 = \xi^2+\eta^2$ and $\theta = \arctan\left(\frac{\eta}{\xi}\right)$. Thus, the origin is a center with the periodic orbits inside the circles $\xi^2+\eta^2 = r^2$. Therefore, the period of the periodic orbit of (6) inside the circle of radius r is given by

$$\hat{T}(r) = \frac{2\pi}{f(r^2)}. \tag{10}$$

Let $E > 0$ such that $H(\xi,\eta) = F(\xi^2+\eta^2) = E$. Since $F'(0) = f(0) > 0$, F has an analytic inverse in a neighborhood of zero. Therefore, $r^2 = \xi^2+\eta^2 = F^{-1}(E)$ in a neighborhood of zero. Substituting $r = \sqrt{F^{-1}(E)}$ into (10), we obtain

$$T(E) = \hat{T}(\sqrt{F^{-1}(E)}) = \frac{2\pi}{f(F^{-1}(E))} \overset{(7)}{=} \frac{\pi}{F'(F^{-1}(E))}. \tag{11}$$

Note that, by definition, $T(E)$ is analytic in a neighborhood of zero with $T(0) = \pi/F'(0) > 0$.

Lemma 3.3 *Let $T(E)$ be the analytic function defined in (11). Then there exists an analytic function V, defined in a neighborhood of zero, such that $V(0) = V'(0) = 0$ and $V''(0) > 0$. Moreover, the period function of the Hamiltonian vector field, defined in a neighborhood of $(0,0)$, by*

$$Y(x,y) = y\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial y}, \tag{12}$$

equals $T(E)$.

Proof: Since $F^{-1}(E)$ is analytic in a neighborhood of zero, with $F^{-1}(0) = 0$, there exists a sequence of real numbers $(a_n)_{n\geq 1}$ such that

$$F^{-1}(E) = \sum_{n\geq 1} a_n E^n. \tag{13}$$

Therefore,

$$\left(F^{-1}\right)'(E) = \sum_{n\geq 1} n a_n E^{n-1}. \tag{14}$$

Since $\left(F^{-1}\right)'(0) = 1/F'(0)$, we have that $a_1 = 1/F'(0) > 0$. Let $(b_n)_{n\geq 1})$ be the sequence of real numbers defined by

$$b_n = \frac{n\sqrt{2\pi}\Gamma(n)}{4\Gamma(n+1/2)} a_n, \tag{15}$$

where Γ is Euler's gamma function. Since $\lim\limits_{n\to\infty} \frac{\Gamma(n)}{\Gamma(n+1/2)} = 0$, there exists a constant $M > 0$ such that $|b_n| \leq nM|a_n|$. Therefore, the function $\varphi(E)$ defined by

$$\varphi(z) = \sum_{n\geq 1} b_n z^{2n-1} \tag{16}$$

is analytic in a neighborhood of zero, with $\varphi(0) = 0$ and $\varphi'(0) = b_1 = \sqrt{2}a_1/2 > 0$. Let $\zeta(E)$ be defined in a neighborhood of zero by

$$\zeta(E) = \varphi(\sqrt{E}) = \sum_{n\geq 1} b_n E^{(2n-1)/2}. \tag{17}$$

Since

$$\zeta'(E) = \frac{\varphi'(E)}{2\sqrt{E}},$$

it follows that $\lim_{E\to 0^+} \zeta'(E) = +\infty$. Consequently, $\zeta(E)$ is invertible in a neighborhood of zero. Let V be the function defined in a neighborhood of zero by $V(\tilde{x}) = \zeta^{-1}(\tilde{x})$. By construction, V is analytic in a neighborhood of zero with $V(0) = V'(0) = 0$ and $V''(0) > 0$.

Indeed, since $\varphi'(0) > 0$, the function $\varphi(z)$ is invertible in a neighborhood of zero. Therefore, $x = \zeta(E) = \varphi(\sqrt{E})$ implies that $\zeta^{-1}(x) = [\varphi^{-1}(x)]^2$. It follows that $V(x) = \zeta^{-1}(x) = [\varphi^{-1}(x)]^2$ is analytic in a neighborhood of zero. Moreover, from the definition of V, we have $V(0) = V'(0) = 0$ and

$$V''(0) = 2(\varphi^{-1})'(0) = \frac{2}{\varphi'(0)} = \frac{2}{b_1} > 0.$$

Note that, by definition, V is an even function. Let Y be the Hamiltonian vector field defined by

$$Y(x, y) = y\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial y}. \tag{18}$$

By construction, Y is analytic in a neighborhood of $(0, 0)$ and has a non-degenerate center at $(0, 0)$. Therefore, the periodic orbits of (18) are contained in the level curves $H(x, y) = E$, where H is the Hamiltonian function defined in a neighborhood of $(0, 0)$ by

$$H(x, y) = \frac{y^2}{2} + V(x). \tag{19}$$

Let $\hat{T}(E)$ be the period of the periodic orbit with $H(x, y) = E$ of the vector field (18). By (19) we have that

$$y = \pm\sqrt{2(E - V(x))}. \tag{20}$$

Then

$$\begin{aligned} \hat{T} &= 2\int_{V_-^{-1}(E)}^{V_+^{-1}(E)} \frac{dx}{y} = 2\int_{V_-^{-1}(E)}^{V_+^{-1}(E)} \frac{dx}{\sqrt{2(E - V(x))}} \\ &= \sqrt{2}\int_0^{V_+^{-1}(E)} \frac{dx}{\sqrt{E - V(x)}} - \sqrt{2}\int_0^{V_-^{-1}(E)} \frac{dx}{\sqrt{E - V(x)}}, \end{aligned}$$

where V_-^{-1} and V_+^{-1} denote the inverse of V in $x < 0$ and $x > 0$ respectively. The change of coordinates $x = V_+^{-1}(E)$ and $x = V_-^{-1}(E)$ in the first and second integral above respectively yield

$$\hat{T}(E) = \sqrt{2}\int_0^E \frac{\left(V_+^{-1}(s) - V_-^{-1}(s)\right)' ds}{\sqrt{E - s}}. \tag{21}$$

Since V is even, we have that $V_-^{-1}(E) = -V_+^{-1}(E)$ and, consequently, $V_+^{-1}(E) - V_-^{-1}(E) = 2V_+^{-1}(E) = 2V^{-1}(E)$. Therefore, the period function $\hat{T}(E)$ of (18) parametrized by the energy levels $H = E$ is the analytic function defined by

$$\hat{T}(E) = 2\sqrt{2}\int_0^E \frac{(V^{-1})'(s)ds}{\sqrt{E - s}}. \tag{22}$$

By definition,

$$V^{-1}(E) = \varphi(\sqrt{E}) \overset{(17)}{=} \sum_{n\geq 1} b_n E^{(2n-1)/2}$$

and, therefore,

$$(V^{-1})'(E) = \sum_{n\geq 2} \frac{(2n-1)}{2} b_n E^{(2n-3)/2}.$$

Substituting the series of $(V^{-1})'(E)$ into (22), we obtain

$$\begin{aligned}
\hat{T}(E) &= 2\sqrt{2}\sum_{n\geq 2}\frac{(2n-1)}{2}b_n\int_0^E \frac{s^{(2n-3)/2}ds}{\sqrt{(E-s)}} = 2\sqrt{2}\sum_{n\geq 2}\frac{(2n-1)}{2}b_n\int_0^1 \frac{(Et)^{(2n-3)/2}E dt}{\sqrt{(E-Et)}} \\
&= 2\sqrt{2}\sum_{n\geq 2}\frac{(2n-1)}{2}b_n E^{n-1}\int_0^1 \frac{t^{(2n-3)/2}dt}{\sqrt{(1-t)}} = 2\sqrt{2}\sum_{n\geq 1}\frac{(2n-1)}{2}b_n E^{n-1} 2\frac{\sqrt{\pi}\Gamma(n-1/2)}{\Gamma(n)} \\
&= 2\sqrt{2}\sum_{n\geq 1} b_n \frac{\sqrt{\pi}\Gamma(n+1/2)}{\Gamma(n)}E^{n-1}.
\end{aligned}$$

By definition, $b_n = \frac{n\sqrt{2\pi}\Gamma(n)}{4\Gamma(n+1/2)}a_n$. Then

$$b_n\frac{\sqrt{\pi}\Gamma(n+1/2)}{\Gamma(n)}E^{n-1} = \frac{n\sqrt{2\pi}\Gamma(n)}{4\Gamma(n+1/2)}a_n \cdot \frac{\sqrt{\pi}\Gamma(n+1/2)}{\Gamma(n)} = \frac{\sqrt{2}\pi n a_n}{4}.$$

Therefore

$$\hat{T}(E) = 2\sqrt{2}\sum_{n\geq 1}\frac{\sqrt{2}\pi n a_n}{4}E^{n-1} = \pi\sum_{n\geq 1} n a_n E^{n-1} \overset{(14)}{=} \pi (F^{-1})'(E) = \frac{\pi}{F'(F^{-1}(E))} \overset{(8)}{=} T(E).$$

3.1 Conclusion of the proof of Theorem 2.1

By construction, the vector field (18) has a non-degenerate center at $(0,0)$. Then, by Lemma 3.1, (18) is analytically conjugate, in a neighborhood of $(0,0)$, to the vector field

$$Y(\xi,\eta) = g(\xi^2+\eta^2)\left(\eta\frac{\partial}{\partial\xi} - \xi\frac{\partial}{\partial\eta}\right), \tag{23}$$

where g is an analytic function defined in a neighborhood of the origin such that $g(0) > 0$.

By Lemma 3.2, the period function $\hat{T}(E)$ of (23), parametrized by $H(\xi,\eta) = E$, is the analytic function given by

$$\hat{T}(E) = \frac{\pi}{G'(G^{-1}(E))} = \pi\frac{d}{dE}G^{-1}(E), \tag{24}$$

where G is the analytic function defined by

$$G(z) = \frac{1}{2}\int_0^z g(s)\,ds. \tag{25}$$

By construction, $\hat{T}(E) = T(E) = \pi\frac{d}{dE}F^{-1}(E)$. Therefore, $\pi\frac{d}{dE}G^{-1}(E) = \pi\frac{d}{dE}F^{-1}(E)$. Since $G^{-1}(0) = F^{-1}(0) = 0$, it follows that

$$G^{-1}(E) = \int_0^E \frac{d}{dE}G^{-1}(s)\,ds = \int_0^E \frac{d}{dE}F^{-1}(s)\,ds = F^{-1}(E).$$

Thus, G coincides with F in a neighborhood of zero. This implies that the vector fields (6) and (23) coincide in a neighborhood of $(0,0)$.

Let h_1 and h_2 be the analytic diffeomorphisms that conjugate the fields X and Y (defined in (18)) with (6), respectively. Then the analytic diffeomorphism h, defined in a neighborhood of $(0,0)$ by $h = h_2^{-1} \circ h_1$, conjugates the fields X and Y. Therefore, the vector field X in the definition of Theorem 2.1 is analytically conjugate, in a neighborhood of $(0,0)$, to the Hamiltonian vector field defined in (18). With this, we conclude the proof of Theorem 2.1. ■

A linear map $R : \mathbb{R}^2 \to \mathbb{R}^2$ is called a linear involution on $\mathbb{R}^2$ if $R \neq \text{id}$ and $R^2 = \text{id}$. A vector field $X : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is reversible with respect to R or R-reversible in Ω if $R \circ X(q) = -X \circ R(q)$ for all $q \in \Omega$.

Corollary 3.4 *Let $X = P\frac{\partial}{\partial u} + Q\frac{\partial}{\partial v}$ be an analytic vector field in a neighborhood of $(0,0)$ with $X(0,0) = (0,0)$ and $P_x(0,0)Q_y(0,0) - P_y(0,0)Q_x(0,0) > 0$. Suppose that X is R-reversible, with $R(u,v) = (u,-v)$. Then X is analytically conjugate, in a neighborhood of $(0,0)$, to the vector field $Y = y\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial y}$, where V is an analytic function defined in a neighborhood of the origin such that $V(0) = V'(0) = 0$ and $V''(0) > 0$.*

Proof: Under these assumptions, X has a non-degenerate center at $(0,0)$ (see [14], Theorem 1). The conclusion follows from Theorem 2.1.

4 Conclusion

As already mentioned, this work provides a partial answer to a question posed by Chicone in 1987. The result opens avenues for further research in more general cases. The global equivalence between vector fields of the form

$$X(u,v) = v\partial_u + f(u, v^2/2)\partial_v$$

and those of the form

$$Y(x,y) = y\frac{\partial}{\partial x} - V'(x)\frac{\partial}{\partial y}$$

has been studied in [15] and [16]. The global version of Theorem 2.1 is proved in Theorem 1.2 of reference [17].

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