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Artigo de Pesquisa

Teoremas de Tauber para sequências triplas somáveis $(\bar{N}, p, q, w)$ de números fuzzy

Tauberian theorems for $(\bar{N}, p, q, w)$ summable triple sequences of fuzzy numbers

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Abstract

In this paper, we introduce the notion of weighted mean method $(\bar{N}, p, q, w)$ of triple sequences of fuzzy numbers and show necessary and sufficient Tauberian conditions under which convergence in Pringsheim's sense of a triple sequence of fuzzy numbers follows from its $(\bar{N}, p, q, w)$ summability. These conditions are weaker than the weighted analogues of Landau's conditions and Schmidt's slow oscillation condition in some senses for three-dimensional case.

Keywords: triple sequence of fuzzy numbers; weighted mean method; tauberian theorems and conditions.

Resumo

Neste artigo, introduzimos a noção do método da média ponderada $(\bar{N}, p, q, w)$ para sequências triplas de números fuzzy e mostramos condições de Tauber necessárias e suficientes sob as quais a convergência no sentido de Pringsheim de uma sequência tripla de números fuzzy decorre de sua somabilidade $(\bar{N}, p, q, w)$. Essas condições são, em certos aspectos, mais fracas do que os análogos ponderados das condições de Landau e da condição de oscilação lenta de Schmidt no caso tridimensional.

Palavras-chave: sequência tripla de números fuzzy; método da média ponderada; teoremas e condições de Tauber.

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1 Introduction

Let $p = (p_j), q = (q_k)$ and $w = (w_i)$ be three sequences of non-negative numbers $p_0, q_0, w_0 > 0$ with the property that

$$P_m = \sum_{j=0}^{m} p_j \to \infty,\ m \to \infty, \tag{1}$$

$$Q_n = \sum_{k=0}^{n} p_k \to \infty,\ k \to \infty \tag{2}$$

and

$$W_h = \sum_{i=0}^{h} p_i \to \infty,\ i \to \infty, \tag{3}$$

The weighted means of a triple sequence $u = (u_{mns})$ of complex numbers are the $(\bar{N}, p, q, w)$ means t_{mnh}, which are defined by

$$t_{mnh} = \frac{1}{P_m Q_n W_h} \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{i=0}^{h} p_j q_k w_i u_{jki},$$

where m, n and h are non-negative integers.

A triple sequence (u_{mns}) of complex numbers is called $(\bar{N}, p, q, w)$ summable to a finite number s if (t_{mnh}) converges to s, i.e. $u_{mnh} \to s(\bar{N}, p, q, w)$.

We can easily check the fact that every P-convergent and bounded double sequence of complex numbers is $(\bar{N}, p, q, w)$ summable to its P-convergence, but the converse of this implication is not true in general.

The notion of $(C, 1, 1, 1)$ summability of a triple sequence was originally introduced by Canak and Totur (2016). Then, Canak *et al.* (2016) studied some $(C, 1, 1, 1)$ means of a statistical convergent triple sequence and gave some classical Tauberian theorems for a triple sequence that P-convergence follows from statistically $(C, 1, 1, 1)$ summability under the two-sided boundedness conditions and slowly oscillating conditions in certain senses. Later, in Totur and Canak (2020) proved Tauberian conditions under which convergence of triple integrals follows from $(C, 1, 1, 1)$ summability.

The results in this work may find some applications in human trafficking or computational DNA sequencing. To do this, one may use the ideas developed to represent data in Ressom *et al.* (2005) for applications in sequencing with refer to Kim *et al.* (2008) and Acharjee and Mordeson (2017). Our main aim in this paper is to give necessary and sufficient Tauberian conditions under which convergence in Pringsheim's sense of a triple sequence of fuzzy numbers follows from its $(\bar{N}, p, q, w)$ summability. We also prove that these conditions are weaker than the weighted analogues of Landau's conditions and Schmidt's slow oscillation condition in some senses for three-dimensional case.

2 Preliminaries

In this section, We recall some notations and basic definitions which are used throughout this paper.

Definition 2.1 *A fuzzy number is a fuzzy set on $\mathbb{R}$, i.e. a mapping $u : \mathbb{R} \to [0,1]$ with the following properties:*

1. *u is normal, i.e. there is a unique element $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.*
2. *u is fuzzy convex, i.e. for any $x, y \in \mathbb{R}$ and for any $\lambda \in [0,1]$, $\mu(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$.*
3. *u is upper semicontinuous.*
4. *The support of u, $[u_0] = \{x \in \mathbb{R} : \bar{u}(x) > 0\}$ is compact, where $\{x \in \mathbb{R} : \bar{u}(x) > 0\}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.*

Remark 2.1 *We denote the set of all fuzzy numbers on $\mathbb{R}$ by E^1 and call it the space of fuzzy numbers. For $u \in E^1$, the α-level set of u is defined by*

$$[u]_\alpha = \frac{\{x \in \mathbb{R} : u(x) \geq 0\},\ 0 < \alpha \leq 1}{\{x \in \mathbb{R} : u(x) > 0\},\ \alpha = 0}.$$

Remark 2.2 *The set $[u]_\alpha$ is a closed, bounded and non-empty interval for each $\alpha \in [0,1]$ with $[u]_\beta \subset [u]_\alpha$ if $\alpha < \beta$ which is also defined by $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$ (Dubois; Prade, 1980).*

Definition 2.2 *The set of all real numbers can be embedded in E^1 since for each $r \in \mathbb{R}, \bar{r} \in E^1$ is defined by E^1 as*

$$\bar{r}(x) = \begin{cases} 1, & x = r \\ 0, & x \neq r. \end{cases}$$

Lemma 2.1 *Bede (2013)*

1. *The addition of fuzzy number is associative and commutative, i.e. $u + v = v + u$ and $u + (v + w) = (u + v) + w$, for all $u, v, w \in E^1$.*
2. *$\bar{0} \in E^1$ is neutral element with respect to $+$, i.e. $u + \bar{0} = \bar{0} + u = u$, for any $u \in E^1$.*
3. *With respect to $+$, none of $u \in E^1 - \mathbb{R}$ has opposite in E^{-1}.*
4. *For any $a, b \in \mathbb{R}$ with $ab \geq 0$ and any $u \in E^1$, we have $(a + b)u = au + bu$. For general $a, b \in \mathbb{R}$ this property does not hold.*
5. *For any $a \in \mathbb{R}$ and $u, v \in E^1$, we have $a(u + v) = au + av$.*
6. *For any $a, b \in \mathbb{R}$ and any $u \in E^1$, we have $(ab)u = a(bu)$.*

Definition 2.3 *Let W be the set of all closed and bounded intervals A of real numbers with endpoints A^- and A^+, i.e., $A = [A^-, A^+]$. The Hausdorff distance on W is defined by*

$$d(A, B) = \max\{|A^- - B^-|, |A^+ - B^+|\}.$$

Remark 2.3 *It is well-known that with respect to the Hausdorff distance, W is a complete metric space (Nanda, 1989). Now, we can define the metric D on the space of fuzzy numbers by means of the Hausdorff metric d.*

Definition 2.4 *Bede (2013), let $D : E^1 \times E^1 \to \mathbb{R}_+$,*

$$\begin{aligned} D(u,v) &= \sup_{\alpha\in[0,1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\} \\ &= \sup_{\alpha\in[0,1]} d([u]_\alpha, [v]_\alpha). \end{aligned}$$

Then, D is said to be Hausdorff distance between fuzzy numbers u and v.

Remark 2.4 *The additive identity and multiplicative identity in E^1 are denoted by $\bar{0}$ and $\bar{1}$, respectively.*

Proposition 2.1 *Bede (2013), let $u, v, , w, z \in E^1$ and $k \in \mathbb{R}$. Then, the following statements hold:*

1. *(E^1, D) is a complete metric space.*
2. *$D(u + w, v + w) = D(u, v)$, i.e. D is translation invariant.*
3. *$D(ku, kv) = |k|D(u, v)$.*
4. *$D(u + v, w + z) \leq D(u, w) + D(v, z)$.*
5. *$|D(u, \bar{0}) - D(v, \bar{0})| \leq D(u, v) \leq D(u, \bar{0}) + D(v, \bar{0})$.*

Now, we will recall some definitions related to triple sequences of fuzzy numbers.

Definition 2.5 *A triple sequence $u = (u_{mnh})$ of fuzzy numbers is a function u from $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ ($\mathbb{N}$ denotes the set of all natural numbers) into the set E^1. The fuzzy number u_{mnh}) denotes the value of the function at a point $(m, n, h) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and is called the (m, n, h)-term of the triple sequence.*

Remark 2.5 *By $w^3(F)$, we denote the set of all triple sequences of fuzzy numbers.*

Definition 2.6 *A triple sequence $u = (u_{mnh})$ of fuzzy numbers is said to be bounded if there exists a positive number M such that $D(u_{mnh}, \bar{0}) < M$ for all m, n and h, i.e. if $\|u\|_{\infty,3} = \sup_{m,n,h} D(u_{mnh}, \bar{0}) < \infty$.*

Remark 2.6 *We will denote the set of all bounded triple sequences of fuzzy numbers by $\ell^3_\infty(F)$.*

Remark 2.7 *In section 4 part (1), we show that every P-convergent triple sequences of fuzzy numbers need not be bounded.*

Definition 2.7 *A triple sequence $u = (u_{mnh})$ of fuzzy numbers is said to be convergent in Pringsheim's sense (or P-convergent) to the fuzzy number μ_0, written as $\lim_{m,n,h\to\infty} u_{mnh} = \mu_0$, if for every $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that $D(u_{mnh}, \mu_0) < \varepsilon$ whenever $m, n, h \geq n_0$, and we denote by P-$\lim u = \mu_0$. The number μ_0 is called the Pringsheim limit of u.*

Remark 2.8 *By $c^3(F)$, we will denote the set of all convergent triple sequences of fuzzy numbers.*

Definition 2.8 *Let $p = (p_j), q = (q_k)$ and $w = (w_i)$) be three sequences of non-negative numbers ($p_0, q_0, w_0 > 0$) with the property (1), (2) and (3). The weighted means of a triple sequence $u = (u_{mnh})$ of fuzzy numbers are the $(\bar{N}, p, q, w)$ means t_{mnh}, which are defined by*

$$t_{mnh} = \frac{1}{P_m Q_n W_h} \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{i=0}^{h} p_j q_k w_i u_{jki},$$

where m, n and h are non-negative integers.

Definition 2.9 *A triple sequence (u_{mnh}) of fuzzy numbers is said to be $(\bar{N}, p, q, w)$ summable to a fuzzy number μ_0 if*

$$\lim_{m,n,h\to\infty} t_{mnh} = \mu_0. \tag{4}$$

Remark 2.9 *Note that throughout this paper we always mean convergence in Pringsheim's sense (Zygumnd, 1959).*

3 Main Results

In this section, we state that $(\bar{N}, p, q, w)$ summability method is a regular method for bounded triple sequences of fuzzy numbers.

Lemma 3.1 *1. If $\lambda > 1, \lambda_m > m, \lambda_n > n$ and $\lambda_h > h$, then*

$$\begin{aligned}
D(\tau^{>}_{mnh}, t_{mnh}) &\leq \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} D(t_{\lambda_m,\lambda_n,\lambda_h}, t_{\lambda_m,n,h}) \\
&+ \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} D(t_{mnh}, t_{m,\lambda_n,h}) \\
&+ \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} D(t_{mnh}, t_{m,n,\lambda_h}) \\
&+ \frac{P_{\lambda_m}}{P_{\lambda_m} - P_m} D(t_{\lambda_m,n,h}, t_{mnh}) + \frac{Q_{\lambda_n}}{Q_{\lambda_n} - P_n} D(t_{m,\lambda_n,h}, t_{mnh}) \\
&+ \frac{W_{\lambda_h}}{W_{\lambda_h} - W_h} D(t_{m,n,\lambda_h}, t_{mnh}).
\end{aligned}$$

2. If $0 < \lambda < 1, \lambda_m < m, \lambda_n < n$ and $\lambda_h < h$, then

$$\begin{aligned}
D(\tau^{<}_{mnh}, t_{mnh}) &\leq \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} D(t_{\lambda_m,\lambda_n,\lambda_h}, t_{\lambda_m,n,h}) \\
&+ \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} D(t_{mnh}, t_{m,\lambda_n,h})
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} D(t_{mnh}, t_{m,n,\lambda_h}) \\
&+ \frac{P_{\lambda_m}}{P_{\lambda_m} - P_m} D(t_{mnh}, t_{\lambda_m,n,h}) + \frac{Q_{\lambda_n}}{Q_{\lambda_n} - P_n} D(t_{mnh}, t_{m,\lambda_n,h}) \\
&+ \frac{W_{\lambda_h}}{W_{\lambda_h} - W_h} D(t_{mnh}, t_{m,n,\lambda_h}).
\end{aligned}
$$

1. By definition, we have

$$
\begin{aligned}
D(\tau^{>}_{mnh}, t_{mnh}) &= D\Big(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - Q_h)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \sum_{i=h+1}^{\lambda_h} p_j q_k w_i u_{jki} \\
&+ \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - Q_h)} \Big(\sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \sum_{i=h+1}^{\lambda_h} \\
&+ \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \sum_{i=0}^{h} + \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{i=0}^{h}\Big) p_j q_k w_i u_{jki}, t_{mnh} \\
&+ \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - Q_h)} \Big(\sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \sum_{i=h+1}^{\lambda_h} \\
&+ \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \sum_{i=0}^{h} + \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{i=0}^{h}\Big) p_j q_k w_i u_{jki}\Big) \\
&= D\Big(\frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - Q_h)} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} \sum_{i=0}^{\lambda_h} p_j q_k w_i \\
&+ \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - Q_h)} \\
&\sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{i=0}^{h} p_j q_k w_i u_{jki}, t_{mnh} \\
&+ \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - Q_h)} \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} \sum_{i=0}^{\lambda_h} p_j q_k w_i u_{jki} \\
&+ \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - Q_h)} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{n} \sum_{i=0}^{h} p_j q_k w_i u_{jki}\Big) \\
&= D\Big(\frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} t_{\lambda_m,\lambda_n,\lambda_h} \\
&+ \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} t_{mnh}, t_{mnh} \\
&+ \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} t_{m,\lambda_n,\lambda_h} \\
&+ \frac{P_{\lambda_m} Q_{\lambda_n} W_{\lambda_h}}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} t_{\lambda_m,n,h}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= D(\frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}t_{\lambda_m,\lambda_n,\lambda_h}\\
&+\frac{P_mQ_nW_h}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}t_{mnh}\\
&+\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}t_{\lambda_m,n,h}+\frac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}t_{m,\lambda_n,h}+\frac{W_{\lambda_h}}{W_{\lambda_h}-W_h}t_{m,n,\lambda_h},t_{mnh}\\
&+\frac{(P_{\lambda_m}+1)(Q_{\lambda_n}+1)(W_{\lambda_h}+1)}{(P_{\lambda_m}-m)(Q_{\lambda_n}-n)(W_{\lambda_h}-h)}(t_{P_{\lambda_m},n,h}\\
&+t_{m,Q_{\lambda_n},h}+t_{m,n,W_{\lambda_h}}))\\
&\le \frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}D(t_{\lambda_m,\lambda_n,\lambda_h},t_{\lambda_m,n,h})
\end{aligned}
$$

$$
\begin{aligned}
&+D(\frac{P_mQ_nW_h}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}t_{mnh}\\
&+\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}t_{\lambda_m,n,h}+\frac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}t_{m,\lambda_n,h}+\frac{W_{\lambda_h}}{W_{\lambda_h}-W_h}t_{m,n,\lambda_h}+\frac{P_m}{P_{\lambda_m}-P_m}t_{m,n,h}\\
&+\frac{Q_n}{Q_{\lambda_n}-Q_n}t_{m,n,h}+\frac{W_h}{W_{\lambda_h}-W_h}t_{m,n,h}\\
&+t_{mnh},t_{mnh}+\frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}t_{m,\lambda_n,\lambda_h}+\frac{P_m}{P_{\lambda_m}-P_m}t_{m,n,h}\\
&+\frac{Q_n}{Q_{\lambda_n}-Q_n}t_{m,n,h}+\frac{W_h}{W_{\lambda_h}-W_h}t_{mnh}\\
&+t_{mnh})\\
&\le \frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}D(t_{\lambda_m,\lambda_n,\lambda_h},t_{\lambda_m,n,h})\\
&+D(\frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}t_{mnh}\\
&+\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}t_{\lambda_m,n,h}+\frac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}t_{m,\lambda_n,h}+\frac{W_{\lambda_h}}{W_{\lambda_h}-W_h}t_{m,n,\lambda_h}\\
&+\frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}t_{m,\lambda_n,\lambda_h}\\
&+\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}t_{mnh}+\frac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}t_{mnh}+\frac{W_{\lambda_h}}{W_{\lambda_h}-W_h}t_{mnh})
\end{aligned}
$$

then,

$$
\begin{aligned}
D(\tau^{>}_{mnh},t_{mnh}) &\le \frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}D(t_{\lambda_m,\lambda_n,\lambda_h},t_{\lambda_m,n,h})\\
&+\frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}D(t_{mnh},t_{m,\lambda_n,h})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}D(t_{mnh},t_{m,n,\lambda_h})\\
&+\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}D(t_{\lambda_m,n,h},t_{mnh})+\frac{Q_{\lambda_n}}{Q_{\lambda_n}-P_n}D(t_{m,\lambda_n,h},t_{mnh})\\
&+\frac{W_{\lambda_h}}{W_{\lambda_h}-W_h}D(t_{m,n,\lambda_h},t_{mnh}).
\end{aligned}
$$

2. This proof is made similar to part (1).

Remark 3.1 *The condition (14) and the conditions*

$$
\limsup_{m\to\infty}\frac{P_{\lambda_m}}{P_m}<1,\limsup_{n\to\infty}\frac{Q_{\lambda_n}}{Q_n}<1 \text{ and } \limsup_{h\to\infty}\frac{W_{\lambda_h}}{W_h}<1 \tag{5}
$$

for every $0<\lambda<1$ *are equivalent (Chen;Hsu, 2000).*

Lemma 3.2 *Let* $p=(p_j), q=(q_k)$ *and* $w=(w_i)$ *be sequences of non-negative numbers such that* $p_0,q_0,w_0>0$ *and condition (14) is satisfied for every* $\lambda>1$. *If* (u_{mnh}) *is* $(\bar{N},p,q,w)$ *summable to a fuzzy number* u_0*, then*

$$
\lim_{m,n,h\to\infty}\tau^{>}_{mnh}=\mu_0 \tag{6}
$$

and

$$
\lim_{m,n,h\to\infty}\tau^{<}_{mnh}=\mu_0 \tag{7}
$$

By Proposition 2.1 parts (2) and (4), we have

$$
\begin{aligned}
D(\tau^{>}_{mnh},\mu_0)&=D(\tau^{>}_{mnh}+t_{mnh},t_{mnh}+\mu_0)\\
&D(\tau^{>}_{mnh},t_{mnh})+D(t_{mnh},\mu_0).
\end{aligned} \tag{8}
$$

By Lemma 3.1 part (1), and (8), we obtain

$$
\begin{aligned}
D(\tau^{>}_{mnh},\mu_0)\le&\frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}D(t_{\lambda_m,\lambda_n,\lambda_h},t_{\lambda_m,n,h})\\
&+\frac{P_{\lambda_m}Q_{\lambda_n}W_{\lambda_h}}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)(W_{\lambda_h}-W_h)}D(t_{mnh},t_{m,\lambda_n,\lambda_h})+\\
&+\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}D(t_{\lambda_m,n,h},t_{mnh})+\frac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}D(t_{m,\lambda_n,h},t_{mnh})\\
&+\frac{W_{\lambda_h}}{W_{\lambda_h}-P_h}D(t_{m,n,\lambda_h},t_{mnh})+D(t_{mnh},\mu_0).
\end{aligned} \tag{9}
$$

For all $\lambda>1$ and sufficiently large m,n,h we get

$$
\limsup_{m\to\infty}\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}=(1-\limsup_{m\to\infty}\frac{P_m}{P_{\lambda_m}})^{-1}<\infty \tag{10}
$$

$$
\limsup_{n\to\infty}\frac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}=(1-\limsup_{n\to\infty}\frac{Q_n}{Q_{\lambda_n}})^{-1}<\infty \tag{11}
$$

$$\limsup_{h\to\infty} \frac{W_{\lambda_h}}{W_{\lambda_h} - W_h} = (1 - \limsup_{h\to\infty} \frac{W_h}{W_{\lambda_h}})^{-1} < \infty \tag{12}$$

from (4). The inequalities (10), (11) and (12) hold for λ_m with sufficiently large m, λ_n with sufficiently large n and λ_h with sufficiently large h. Since (u_{mnh}) is $(\bar{N}, p, q, w)$ summable to a fuzzy number μ_0 and (10), (11) and (12) hold, it is clear that the terms on the right of (9) vanish under the operator $\limsup_{m,n,h\to\infty}$. Therefore, (6) follows from (8).

The proof of (7) is made similar.

Theorem 3.1 *If $u = (u_{mnh}) \in w^3(F) \cap \ell^3_\infty(F)$ is convergent to $\mu_0 \in E^1$, then (u_{mnh}) is $(\bar{N}, p, q, w)$ summbale to μ_0.*

Let $(u_{mnh}) \in c^3(F) \cap \ell^3_\infty(F)$. Then, there exists $\mu_0 \in E^1$ such that $\lim_{m,n,h\to\infty} D(u_{mnh}, \mu_0) = 0$. Therefore, by Proposition 2.1 part (4), we have

$$\begin{aligned} D(t_{mnh}), \mu_0) &= D(\frac{1}{P_m Q_n W_h} \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{i=0}^{h} p_j q_k w_i u_{jki}, \mu_0) \\ &\leq \frac{1}{P_m Q_n W_h} \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{i=0}^{h} p_j q_k w_i D(u_{jki}, \mu_0). \end{aligned}$$

Since $\lim_{m,n,h\to\infty} D(u_{mnh}, \mu_0) = 0$, by the regularity of $(\bar{N}, p, q, w)$ summability method and the boundedness (u_{mnh}) of sequences of real numbers, we get

$$\lim_{m,n,h\to\infty} D(t_{mnh}, \mu_0) = 0. \tag{13}$$

By the following example, we show that the converse of Theorem 3.1 does not hold in general.

Example 3.1 *Consider the triple sequence $u = (u_{mnh})$ of fuzzy numbers defined by*

$$u_{mnh} = \begin{cases} u_0 & if \quad m, n, h \ are \ even \\ v_0 & otherwise, \end{cases}$$

where

$$u_0(t) = \begin{cases} 1 - t & if \quad t \in [0, 1] \\ 0 & otherwise \end{cases}$$

and

$$v_0(t) = \begin{cases} 1 + t & if \quad t \in [-1, 0] \\ 0 & otherwise. \end{cases}$$

It is clear that (u_{mnh}) is divergent. The endpoints of the α-level set of the triple sequence (u_{mnh}) of fuzzy numbers are

$$u^{-}_{mnh} = \begin{cases} 0 & if \quad m,n,h \; are \; even \\ \\ \alpha - 1 & otherwise \end{cases}$$

and

$$u^{+}_{mnh} = \begin{cases} 1-\alpha & if \quad m,n,h \; are \; even \\ \\ 0 & otherwise. \end{cases}$$

For $p = q = w = 1$, the triple sequence $(t^{-}_{mnh}(\alpha))$ and $(t^{+}_{mnh}(\alpha))$ converge to $\frac{3(\alpha-1)}{4}$ and $\frac{(1-\alpha)}{4}$ as $m,n,h \to \infty$, respectively. We have $\lim_{m,n,h\to\infty} D(t_{mnh}, \omega_0) = 0$, where $\omega_0 = \frac{(u_0+3v_0}{4}$. In others words, the triple sequence (t_{mnh}) of fuzzy number is convergent to ω_0 and then it is $(\bar{N}, 1, 1, 1)$ summblae to ω_0.

Throughout this paper, λ_n presents the integral part of the product λn, in symbol $\lambda_n = [\lambda n]$. The next result is a partial converse of Theorem 3.1.

Theorem 3.2 *Let $p = (p_j), q = (q_k)$ and $w = (w_i)$ be sequences of non-negative numbers with $p_0, q_0, w_0 > 0$ and*

$$\limsup_{m\to\infty} \frac{P_m}{P_{\lambda_m}} < 1, \limsup_{n\to\infty} \frac{Q_n}{Q_{\lambda_n}} < 1 \; and \; \limsup_{h\to\infty} \frac{W_h}{W_{\lambda_h}} < 1 \tag{14}$$

for every $\lambda > 1$.
If (u_{mnh}) is $(\bar{N}, p, q, w)$ summable to a fuzzy number μ_0, then the limit

$$\lim_{m,n,h\to\infty} u_{mnh} = \mu_0 \tag{15}$$

exists if and only if

$$\liminf_{\lambda\to 1} \limsup_{m,n,h\to\infty} D(\tau^{>}_{mnh}, u_{mnh}) = 0 \tag{16}$$

or

$$\liminf_{\lambda\to 1} \limsup_{m,n,h\to\infty} D(\tau^{<}_{mnh}, u_{mnh}) = 0, \tag{17}$$

where, for $\lambda > 1$

$$\tau^{>}_{mnh} = \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \sum_{i=h+1}^{\lambda_h} p_j q_k w_i u_{jki}$$

and for $0 < \lambda < 1$

$$\tau^{<}_{mnh} = \frac{1}{(P_m - P_{\lambda_m})(Q_n - Q_{\lambda_n})(W_h - W_{\lambda_h})} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} \sum_{i=\lambda_h+1}^{h} p_j q_k w_i u_{jki}$$

for sufficiently large non-negative integers m, n, h.

Necessity: If (u_{mnh}) is $(\bar{N}, p, q, w)$ summable to μ_0 and (15) is satisfied, then

$$\lim_{m,n,h\to\infty} D(u_{mnh}, t_{mnh}) = 0. \tag{18}$$

Now, (16) (resp. (17)) follows from (6) (resp. (7)) and (18).

Sufficiency: Consider that condition (16) holds. Then, for any given $\varepsilon > 0$ there exists $\lambda_0 > 1$ such that

$$\limsup_{m,n,h\to\infty} D(\tau^{>}_{mnh}, u_{mnh}) \leq \varepsilon. \tag{19}$$

Besides, we have

$$\begin{aligned} D(u_{mnh}, \mu_0) &= D(u_{mnh} + \tau^{>}_{mnh}, \mu_0 + \tau^{>}_{mnh}) \\ &\leq D(u_{mnh}, \tau^{>}_{mnh}) + D(\tau^{>}_{mnh}, \mu_0) \end{aligned}$$

by Proposition 2.1 parts (2) and (4). Taking into account that (u_{mnh}) is $(\bar{N}, p, q, w)$ summbale to μ_0, (6) and (19), we obtain

$$\limsup_{m,n,h\to\infty} D(u_{mnh}, \mu_0) \leq \varepsilon. \tag{20}$$

In a similar way, if (17) holds, for any given $\varepsilon > 0$ there exists $0 < \lambda_0 < 1$ such that

$$\limsup_{m,n,h\to\infty} D(\tau^{<}_{mnh}, u_{mnh}) \leq \varepsilon. \tag{21}$$

Furthermore, we have

$$\begin{aligned} D(u_{mnh}, \mu_0) &= D(u_{mnh} + \tau^{<}_{mnh}, \mu_0 + \tau^{<}_{mnh}) \\ &\leq D(u_{mnh}, \tau^{<}_{mnh}) + D(\tau^{<}_{mnh}, \mu_0) \end{aligned}$$

by Proposition 2.1 parts (2) and (4). Taking into account that (u_{mnh}) is $(\bar{N}, p, q, w)$ summable to μ_0, (7) and (21), we obtain

$$\limsup_{m,n,h\to\infty} D(u_{mnh}, \mu_0) \leq \varepsilon. \tag{22}$$

Since ε is arbitrary, this ends the proof.

Definition 3.1 *A triple sequence (u_{mnh}) of fuzzy numbers is said to be slowly oscillating in the strong sense $(1, 1, 1)$ if*

$$\lim_{\lambda\to 1} \limsup_{m,n,h\to\infty} \max_{\underline{m+1,n+1,h+1\leq j,k,i\leq\lambda_m,\lambda_n,\lambda_h}} D(u_{jki}, u_{mnh}) = 0. \tag{23}$$

Equivalently, we say that (u_{mnh}) is slowly oscillating in the sense $(1, 1, 1)$ if for each $\varepsilon > 0$ there exist $n_1(\varepsilon)$ and $\lambda_0 = \lambda_0(\varepsilon) > 1$ such that $D(u_{jki}, u_{mnh}) \leq \varepsilon$ whenever $n_1 < m < j \leq \lambda_m$, $n_1 < n < k \leq \lambda_n$ and $n_1 < h < i \leq \lambda_h$.

Remark 3.2 *By $s^3_{111}(F)$, we will denote the set of all slowly oscillating triple sequences in the sense $(1,1,1)$ of fuzzy numbers.*

As corollary of Theorem 3.2, we have the following result.

Corollary 3.1 *Let (14) be satisfied. If $(u_{mnh}) \in s^3_{111}$ and (u_{mnh}) is $(\bar{N},p,q,w)$ summable to a fuzzy number μ_0, then (u_{mnh}) is convergent to μ_0.*

Consider that (u_{mnh}) is slowly oscillating in the sense $(1,1,1)$. Since

$$D(\tau^{<}_{mnh}, u_{mnh}) \leq \frac{1}{(P_{\lambda_m} - P_m)(Q_{\lambda_n} - Q_n)(W_{\lambda_h} - W_h)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \sum_{i=h+1}^{\lambda_h} p_j q_k w_i D(u_{jki}, u_{mnh})$$
$$\leq \underline{\max_{m+1,n+1,h+1 \leq j,k,i \leq \lambda_m,\lambda_n,\lambda_h}} D(u_{jki}, u_{mnh}).$$

Therefore, we conclude that (16) holds. By Theorem 3.2, (u_{mnh}) is convergent to μ_0.

Definition 3.2 *We say that a double sequence (u_{mnh}) of fuzzy numbers is said to be slowly oscillating in the sense $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ if*

$$\lim_{\lambda \to 1} \limsup_{m,n,h \to \infty} \max_{m+1 \leq j \lambda_m} D(u_{jnh}, u_{mnh}) = 0, \tag{24}$$

$$\lim_{\lambda \to 1} \limsup_{m,n,h \to \infty} \max_{n+1 \leq k \lambda_n} D(u_{mkh}, u_{mnh}) = 0 \tag{25}$$

and

$$\lim_{\lambda \to 1} \limsup_{m,n,h \to \infty} \max_{h+1 \leq i \lambda_h} D(u_{mni}, u_{mnh}) = 0. \tag{26}$$

Respectively.

Remark 3.3 *By s^3_{100}, s^3_{010} and s^3_{001}, we will denote the set of all slowly oscillating triple sequences in the sense $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ of fuzzy numbers, respectively.*

Lemma 3.3 *If $(u_{mnh}) \in s^3_{100}(F) \cap s^3_{010}(F) \cap s^3_{001}(F)$, then $(u_{mnh}) \in s111^3(F)$.*

By the triangle inequality, we have

$$D(u_{jki}, u_{mnh}) \leq D(u_{jki}, u_{mki}) + D(u_{mki}, u_{mnh}). \tag{27}$$

From (27), we conclude that

$$\begin{aligned} \underline{\max_{m+1,n+1,h+1 \leq j,k,i \leq \lambda_m,\lambda_n,\lambda_h}} D(u_{jki}, u_{mnh}) &\leq \underline{\max_{m+1,n+1,h+1 \leq j,k,i \leq \lambda_m,\lambda_n,\lambda_h}} D(u_{jki}, u_{mki}) \\ &+ \underline{\max_{m+1,n+1,h+1 \leq j,k,i \leq \lambda_m,\lambda_n,\lambda_h}} D(u_{mki}, u_{mnh}). \end{aligned} \tag{28}$$

Since (u_{mnh}) is slowly oscillating in the senses $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, it follows from (28) that (u_{mnh}) is slowly oscillating in the sense $(1,1,1)$.

Corollary 3.2 *Let (14) satisfied. If* (u_{mnh}) *is slowly oscillating in the sense* $(1,0,0)$, $(0,1,0)$ *and* $(0,0,1)$ *and* $(\bar{N},p,q,w)$ *summbale to a fuzzy number to* μ_0*, then* (u_{mnh}) *is convergent to* μ_0.

The proof is followed by Lemma 3.3 and Theorem 3.2.

Remark 3.4 *We say that* (u_{mnh}) *satisfies weighted analogue of Landau's condition in sense* $(1,0,0)$ *if there exist* n_1 *and* B *such that*

$$\frac{P_j}{p_j} D(u_{jnh}, u_{j-1,n,h}) \leq B \; whenever \; j,n,h > n_1. \tag{29}$$

It is clear that if (29) holds and P *is regularly varying of positive index, then* $(u_{mnh}) \in s^3_{100}(F)$*. We remind the reader that* P *is regularly varying of index* $\alpha > 0$ *if*

$$\lim_{n\to\infty} \frac{P(\lambda_n)}{P(n) = \lambda^\alpha}, \; \lambda > 0. \tag{30}$$

The relation between sequences satisfying conditions (14) and (30) is discussed in Chen and Hsu (2000) and noted that (14) is clearly satisfied if P *is regularly varying of index* $\alpha > 0$*. We say that* (u_{mnh}) *satisfies weighted analogue of Landau's condition in sense* $(0,1,0)$ *if there exist* m_1 *and* C *such that*

$$\frac{Q_k}{q_k} D(u_{mkh}, u_{m,k-1,h}) \leq C \; whenever \; m,k,h > m_1. \tag{31}$$

It is clear that if (31) holds and P *is regularly varying of positive index, then* $(u_{mnh}) \in s^3_{010}(F)$*.*

Analogous, we say that (u_{mnh}) *satisfies weighted analogue of Landau's condition in sense* $(0,0,1)$ *if there exist* h_1 *and* R *such that*

$$\frac{W_i}{w_i} D(u_{mni}, u_{m,n,i-1}) \leq R \; whenever \; m,n,i > h_1. \tag{32}$$

It is clear that if (32) holds and P *is regularly varying of positive index, then* $(u_{mnh}) \in s^3_{001}(F)$*.*

Lemma 3.4 *If* (u_{mnh}) *satisfies weighted analogues of Landau's condition in senses* $(1,0,0)$, $(0,1,0)$ *and* $(0,0,1)$*, then* $(u_{mnh}) \in s^3_{111}(F)$*.*

Consider that (u_{mnh}) satisfies weighted analogues of Landau's condition in senses $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, i.e. there exist $n_1(\varepsilon)$ and R such that

$$\frac{P_m}{p_m} D(u_{mnh}, u_{m-1,n,h}) \leq R,$$

$$\frac{Q_n}{q_n} D(u_{mnh}, u_{m,n-1,h}) \leq R$$

and

$$\frac{W_h}{w_h} D(u_{mnh}, u_{m,n,h-1}) \leq R$$

whenever $m,n,h > n_1$. For all $1 < n_1 < m < j \leq \lambda_m$, $1 < n_1 < n < k \leq \lambda_n$ and $1 < n_1 < h < i \leq \lambda_h$, we get

$$\begin{aligned} D(u_{jki}, u_{mnh}) &\leq D(u_{jki}, u_{mki}) + D(u_{mki}, u_{mnh}) \\ &\leq \sum_{r=m+1}^{j} D(u_{rnh}, u_{r-1,n,h}) \sum_{s=n+1}^{k} D(u_{msh}, u_{m,s-1,h}) + \sum_{g=h+1}^{i} D(u_{mng}, u_{m,n,g-1}) \\ &\leq H(\sum_{r=m+1}^{j} \frac{p_r}{P_r} + \sum_{s=n+1}^{k} \frac{q_s}{Q_s} + \sum_{g=h+1}^{i} \frac{w_g}{W_g}) \\ &= H(\frac{P_j - P_m}{P_m} + \frac{Q_k - Q_n}{Q_n} + \frac{W_i - W_h}{W_h}) \end{aligned}$$

which implies that

$$\max_{\underline{m+1,n+1,h+1 \leq j,k,i \leq \lambda_m,\lambda_n,\lambda_h}} D(u_{jki}, u_{mnh}) \leq H(\frac{P_{\lambda_m} - P_m}{P_m} + \frac{Q_{\lambda_n} - Q_n}{Q_n} + \frac{W_{\lambda_h} - W_h}{W_h}). \tag{33}$$

Since P, Q and W are regularly varying of index $\alpha, \beta, \gamma > 0$, respectively. It follows from (33) that

$$\limsup_{m,n,h\to\infty} \max_{\underline{m+1,n+1,h+1 \leq j,k,i \leq \lambda_m,\lambda_n,\lambda_h}} D(u_{jki}, u_{mnh}) \leq H(\lambda^{\alpha} + \lambda^{\beta} + \lambda^{\gamma} - 3)$$

which implies that

$$\liminf_{\lambda\to 1} \limsup_{m,n,h\to\infty} \max_{\underline{m+1,n+1,h+1 \leq j,k,i \leq \lambda_m,\lambda_n,\lambda_h}} D(u_{jki}, u_{mnh}) = 0.$$

Corollary 3.3 *Let (14) be satisfied. If* (u_{mnh}) *satisfies weighted analogues of Landau's condition in senses* $(1,0,0)$, $(0,1,0)$ *and* $(0,0,1)$ *and is* $(\bar{N}, p, q, w)$ *summable to a fuzzy number to* μ_0*, then* (u_{mnh}) *is convergent to* μ_0.

The proof is followed by Lemma 3.4 and Theorem 3.2.

4 Some Applications

1. The triple sequence (u_{mnh}) of fuzzy numbers defined by

$$u_{mnh} = \begin{cases} \beta_n & if \quad m = h = 0, n \in \mathbb{N} \\ 0 & otherwise. \end{cases}$$

where $\beta_n = \sum_{k=0}^{n} \gamma_k$ and

$$\gamma_k(t) = \begin{cases} 1 + t(k+1)^2 & if \quad t \in [-\frac{1}{(k+1)^2}, 0] \\ \\ 0 & otherwise. \end{cases}$$

is $(\bar{N}, p, q, w)$ summbale to $\bar{0}$.

As an application of Theorem 3.1. Now, we will show that the triple sequence $u = (u_{mnh})$ is $(\bar{N}, p, q, w)$ summable to $\bar{0}$. It is clear that (u_{mnh}) is P-convergent to $\bar{0}$. The endpoints of the α-level set of the triple sequence $u = (u_{mnh})$ of fuzzy numbers are

$$u^-_{mnh}(\alpha) = \begin{cases} \beta^-_n(\alpha) & if \quad m = h = 0, n \in \mathbb{N} \\ \\ 0 & otherwise. \end{cases}$$

and $u^+_{mnh}(\alpha) = 0$ for all $m, n, h \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \|u\|_{\infty,3} &= \sup_{m,n,h\in\mathbb{N}} D(u_{mnh}, \bar{0}) \\ &= \sup_{m,n,h\in\mathbb{N}} \sup_{\alpha\in[0,1]} \max\{|u^-_{mnh}(\alpha) - 0|, |u^+_{mnh}(\alpha) - 0|\} \\ &= \sup_{m,n,h\in\mathbb{N}} \sup_{\alpha\in[0,1]} |u^-_{mnh}(\alpha)| \\ &= \sup_{m,n,h\in\mathbb{N}} |u^-_{mnh}(0)| \\ &= \sup_{m,n,h\in\mathbb{N}} |\beta^-_n(0)|. \end{aligned}$$

Since the sequence $(\beta^-_n(0)) = (-\sum_{k=0}^{n} \frac{1}{(k+1)^2})$ is convergent as $n \to \infty$, the triple sequence $u = (u_{mnh})$ of fuzzy numbers is bounded. By Theorem 3.1, the triple sequence $u = (u_{mnh})$ is $(\bar{N}, p, q, w)$ summable to $\bar{0}$.

2. The triple sequence $u = (u_{mnh})$ of fuzzy numbers defined by

$$u_{mnh} = \begin{cases} \beta_h & if \quad h \geq n, m \\ \\ \beta_{mn} & otherwise. \end{cases}$$

where

$$\beta_{mn} = \begin{cases} 1 - \frac{t}{1 + \log[(m+1)(n+1)]} & if \quad t \in [0, 1 + \log[(m+1)(n+1)]] \\ \\ 0 & otherwise. \end{cases}$$

is not $(\bar{N}, p, q, w)$ summable.

As an application of Theorem 3.2, we will show that the triple sequence $u = (u_{mnh})$ is not $(\bar{N}, p, q, w)$ summable.

Consider that the triple sequence $u = (u_{mnh})$ is $(\bar{N}, p, q, w)$ summable. Condition (14) is clearly satisfied. One can check that the endpoints of the α-level set of the triple sequence $u = (u_{mnh})$ of fuzzy numbers are $u^-_{mnh}(\alpha) = 0$ for all $m, n, h \in \mathbb{N}$ and

$$u^+_{mnh}(\alpha) = \begin{cases} (1-\alpha)(1+\log(m+1)) & if \quad m \geq h \\ (1-\alpha)(1+\log(n+1)) & if \quad n \geq h \\ (1-\alpha)(1+\log(h+1)) & otherwise. \end{cases}$$

Therefore, we have

$$\max_{\underline{m+1\leq j\leq \lambda_m, n+1\leq k\leq \lambda_n, h+1\leq i\leq \lambda_h}} D(u_{jki}, u_{mnh}) = \max_{\underline{m+1\leq j\leq \lambda_m, n+1\leq k\leq \lambda_n, h+1\leq i\leq \lambda_h}} |u^+_{jki}(0) - u^+_{mnh}(0)|$$

$$= \begin{cases} \log\left(\dfrac{\lambda_{mn}+1}{(m+1)(n+1)}\right) & \text{if} \quad j,k \geq i \text{ and } m,n \geq h, \\ \log\left(\dfrac{\lambda_{mn}+1}{(h+1)}\right) & \text{if} \quad j,k \geq i \text{ and } m,n < h, \\ \log\left(\dfrac{\lambda_h+1}{(m+1)(n+1)}\right) & \text{if} \quad j,k < i \text{ and } m,n \geq h, \\ \log\left(\dfrac{\lambda_h+1}{(h+1)}\right) & \text{if} \quad j,k < i \text{ and } m,n < h. \end{cases}$$

By taking the lim sup of both sides of the last equality as $m, n, h \to \infty$, we have that

$$\limsup_{m,n,h\to\infty} \max_{\underline{m+1\leq j\leq \lambda_m, n+1\leq k\leq \lambda_n, h+1\leq i\leq \lambda_h}} D(u_{jki}, u_{mnh}) = \log \lambda.$$

By taking the limit of both sides of the last equality as $\lambda \to 1$. we arrive that

$$\lim_{\lambda\to 1} \limsup_{m,n,h\to\infty} \max_{\underline{m+1\leq j\leq \lambda_m, n+1\leq k\leq \lambda_n, h+1\leq i\leq \lambda_h}} D(u_{jki}, u_{mnh}) = 0.$$

Therefore, it follows from the definition of the slow oscillation in the sense $(1,1,1)$ that the triple sequence $u = (u_{mnh})$ of fuzzy numbers is slowly oscillating in the sense $(1,1,1)$ and it satisfies condition (16) by Corollary 3.1. By Theorem 3.2, the sequence $u = (u_{mnh})$ is convergent. But this is impossible since (u_{mnh}) is clearly divergent. Thus, the sequence $u = (u_{mnh})$ is not $(\bar{N}, p, q, w)$ summable.

5 Conclusion

In this article, we have given a Tauberian theorem for $(\bar{N}, p, q, w)$ summability of triple sequences of fuzzy numbers as an extension of a Tauberian theorem for sequences of single and double fuzzy sequences due to (Talo; Cakan, 2012) and (Totur; Canak, 2020), respectively. For future works, we suggest studying more Tauberian theorems on $(\bar{N}, p, q, w)$ summability by using triple sequences.

Conflict of interest

The authors declare that they have no conflict of interest.

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