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Artigo de Pesquisa

Algebraic estimation of apothems in regular polygons using linear regression and bootstrap approximation

Estimativa algébrica de apótemas em polígonos regulares pelo uso de regressão linear e aproximação por bootstrap

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Abstract

The estimation of the apothem of regular polygons can be necessary in academic and real-life situations, such as for estimating the value of the area of regular polygons or the volume of prisms and pyramids, and its formal calculation requires the use of the trigonometric tangent function. In the present article, we describe an algebraic approximation procedure for estimating the apothem of any regular polygon given the length of one side and the number of sides. In section 2, we present an initial approximation, obtained based on the almost linear correlation between the product of the number of sides by the side length with the square root of the sum of the apothem squared and the hypotenuse squared of one of the polygon's internal right triangles. In section 3, the bootstrap method was used to obtain the coefficients of a second-degree polynomial. The resulting equation predicts the apothem with a minimum accuracy of 0.9997, and in polygons of 3, 4, 6 or with more than 25 sides, the accuracy was superior to 0.9999.
Keywords: apothem; regular polygon; polygon area.

Resumo

A estimativa do apótema de polígonos regulares pode ser necessária em situações acadêmicas e da vida real, como para estimar o valor da área de polıgonos regulares ou o volume de prismas e pirâmides, e seu cálculo formal requer o uso da função trigonométrica tangente. No presente artigo descrevemos um procedimento de aproximação algébrica para estimativa do apótema de qualquer polígono regular dado o comprimento de um lado e o número de lados. Na seção 2, apresentamos uma aproximação inicial, obtida com base na correlação quase linear entre o produto do número de lados pelo comprimento do lado com a raiz quadrada da soma do apótema elevado ao quadrado e da hipotenusa elevada ao quadrado de um dos triângulos retângulos internos do polígono. Na seção 3, foi utilizado o método bootstrap para obter os coeficientes de um polinômio de segundo grau. A equação resultante prevê o apótema com precisão mıńima de 0,9997, já em polıgonos de 3, 4, 6 ou com mais de 25 lados, a precisão foi superior a 0,9999.
Palavras-chave: apótema; polıgono regular; área poligonal.

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1 Introdução

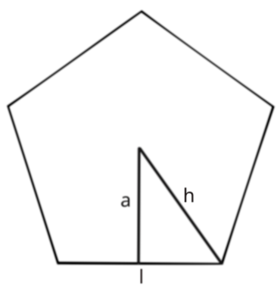
The approximation of non-linear equations has the aim to obtain simple equations that can be used to interpret and understand complex non-linear associations. These procedures are used to analyze or predict the pattern of a distribution of scores or values [1]. Usually, this is not an easy task since, in many cases, an accurate approximation formula cannot produce a general formula obtained for the whole extension of the distribution. In order to overcome these difficulties, many approaches focus on the approximation of specific points or small segments whose distribution can be predicted by a mathematical function [2] [3] [4]. The approximation can also be obtained using the Taylor series, which uses repeated differentiation to produce a polynomial approximation of a function [5]. A more precise approximation can be obtained with Newton's or Padè's approximation methods [6] [7]. However, these procedures may be laborious and complex and may result in polynomials of high degrees.

Estimating the apothem of regular polygons may be required in academic and real-life situations such as in the calculation of the area of regular polygons and the volume of regular prisms and pyramids. Apothem estimation may be necessary to perform simple tasks, such as in the calculation of the number of tiles to cover a surface or in estimating the volume of liquid to fill a prismatic reservoir. Calculation of apothem may also be required in more complex tasks [8–13]. In what follows we will describe an algebraic approximation procedure to estimate the apothem value using computational methods.

2 Approximation using linear regression

For each regular polygon it is possible to obtain 2 n right-angled triangles with external side length of l/2, internal side length, apothem, a and hypotenuse h (Figure 1).

Figure 1: Right-angled triangle inside a regular pentagon.
l = side, a = apothem, h = hypotenuse.



SOURCE: The authors

The area of a regular polygon with n sides ($n \geq 2$) and with side lengths measuring l is given by the formula $A = pr\ a$, where A represents the area, pr the perimeter and a the apothem of the polygon. The apothem is obtained by (1):

$$a = \frac{\ell}{2tan(\pi/n)} \tag{1}$$

In the present article, we describe an approximation procedure for estimation of the apothem of any regular polygon. This procedure produced a simple implementation general algebraic equation that allows the estimation of the apothem of any regular polygon given only the length of one side (l) and the number of sides (n).

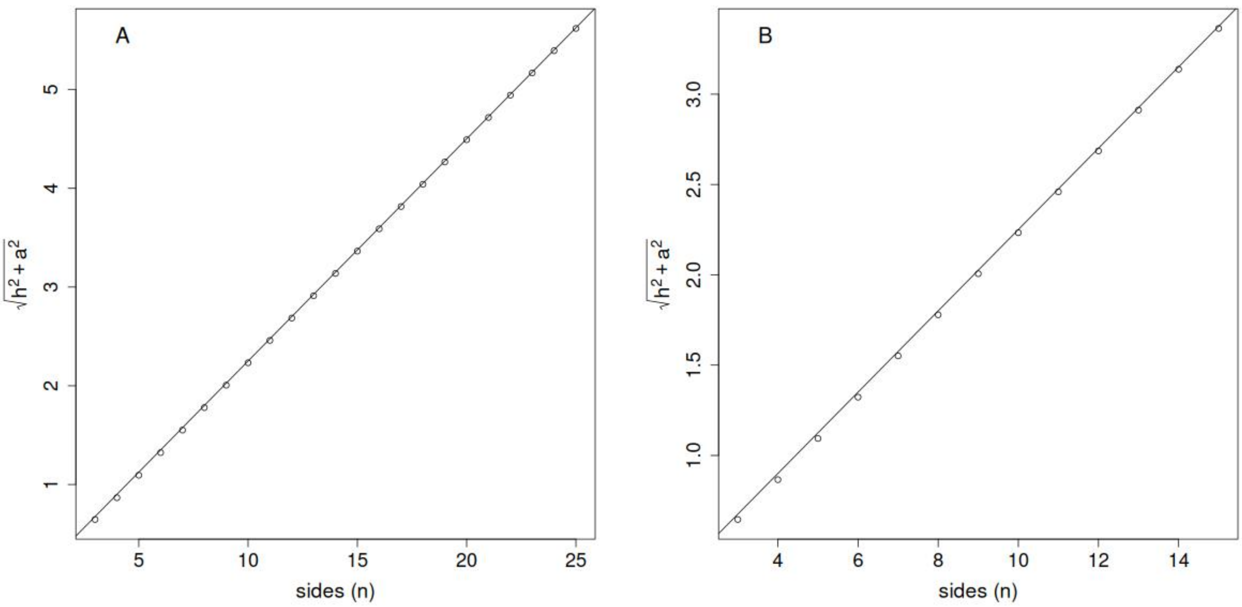
We noted that there is a nearly linear correlation (Coefficient of determination R2 = 1, Pearson R = 0.9999998, p < 2.12e-16, n: 3-100, l =1) between n (independent variable) and $\sqrt{h^2 + a^2}$ (Figure 2). The precision of the prediction of the fitted line equation increases as n increases. Thus, the distribution values in this nearly linear correlation could be approximately represented by (2) or (3).

$$f(n\ell) = \sqrt{h^2 + a^2} \tag{2}$$

$$f(n\ell)^2 = h^2 + a^2 \tag{3}$$

where h and a are positive real numbers that vary for each value of n and represent the hypotenuse and apothem of the triangle represented in Figure 1, respectively.

Figure 2: Dot plots and best-fit lines of (2). A. Polygons with sides varying between 3 and 100. B. Polygons with sides varying between 3 and 10. Note that the polygons with the smallest sides have values slightly smaller than predicted by the linear regression best-fit line. l = side, a = apothem, h = hypotenuse.



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Equation (4) is obtained from Pythagoras' theorem applied in Fig. 1.

$$(\frac{\ell}{2})^2 = h^2 - a^2 \tag{4}$$

The estimated apothem (5) is obtained by applying the substitution method in (3) and (4).

$$ape = \sqrt{\frac{f(n\ell)^2 - (\frac{\ell}{2})^2}{2}} \tag{5}$$

Since *f(nl)* approximates a linear function, it can be estimated using the equation of a straight line:

$$f(n\ell)^2 = intercept + x\,slope \tag{6}$$

To estimate the intercepts and slopes of this function, we performed a series of linear regression analysis with windows of sides x to x+*3* (4 values of x) with increasing values of x up to 100000. It was observed that as x increases, the intercept decreases and tends to zero, and the slope stabilizes at the value 0.225079 *l*. Note that as n increases, the slope increases as a function of *l*. These values were used as regression coefficients in (6).

The numerical values of *ape* can be obtained for each regular polygon by replacing *f(n)* of (6) in (5). Resulting in (7):

$$ape = \ell\sqrt{0.02533\,n^2 - 0.125} \tag{7}$$

The estimated apothems obtained with (7) were compared with apothems calculated using (1). Figure 3A shows the ratio between the estimated apothems (7) and the apothems calculated by (1). As can be seen, the estimated apothems (*ape*) tend to be slightly higher than the values obtained with the trigonometric function (1). In this first step the less precise estimation was obtained for the equilateral triangle ($n = 3$) which was of 0.8996. The precision progressively increased as n increased. Predictions of accuracy equal to or greater than 0.999 were obtained only for polygons with more than 27 sides $n > 28$.

3 Bootstrap polynomial modeling of the estimated apothem

The next step was to obtain an equation to improve the accuracy of the predictions of (7) using the polynomial approximation [14]. As noted in Fig. 3A the values of the estimated apothem (*ape*) were higher than the real apothem (a), and the difference is maximum for the 3-sided polygon, decreasing progressively in a non-linear fashion as the number of sides (n) increases (i.e. non-linear and inversely proportional). Thus, the estimated values obtained with (7) could be further approximated from the apothems obtained by (1) using the general modeling equation (8):

$$apm1 = ape - (ape\frac{1}{k\,n^x}) \tag{8}$$

where k and x are rational numbers ranging from 0 to 5 and n is the number of sides of the regular polygon. The values of k and x were obtained by the Bootstrap method with 1000000 resamplings. The values were selected so the *apm1/ape* ratio produced values between 0.998 and 1.0008 for n up to 100. The aim of this step was to obtain the degree of the polynomial that produced the best approximation. The selected values were $k = 0.8975$ and $x = 2$. Therefore, a polynomial of second degree was chosen for the next step. The approximation formula (9) is:

$$apm1 = \ell\sqrt{0.02533\, n^2 - 0.125}\, (1 - \frac{1}{1.1147\, n^2}) \tag{9}$$

Equation (9) produced a more accurate prediction of regular polygon apothems. For polygons with 5, 6 and 7 sides this formula has an accuracy superior to 0.9987. For all other polygons the formula predicts the apothem with an accuracy higher than 0.999. In polygons with more than 27 sides the accuracy is higher than 0.9999 (Fig. 3B).

Since the best accuracy was obtained with a degree of 2 ($x = 2$) we seek to further increase the accuracy of the predictions using the general form of a quadratic equation [11] (10).

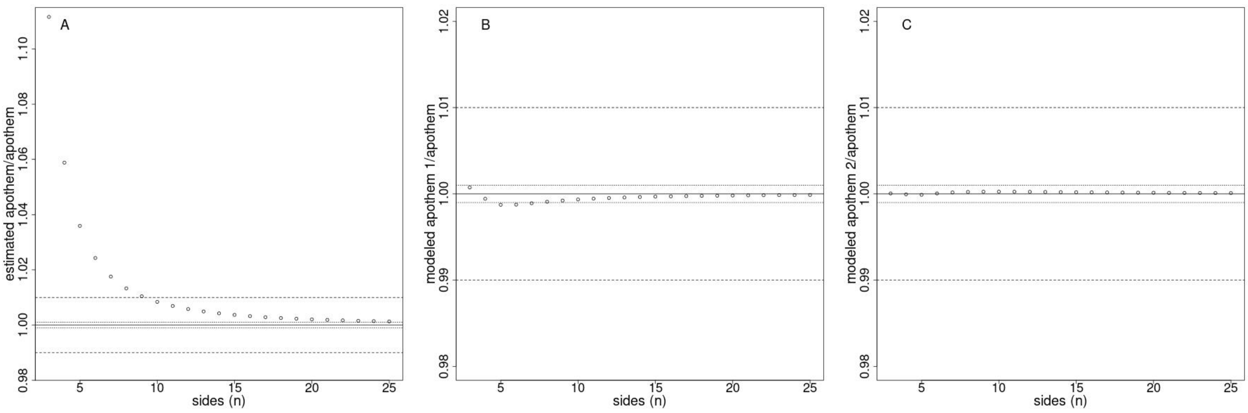
$$apm2 = ape - ape(\frac{1}{b\, n^2 + c\, x + d}) \tag{10}$$

We performed 1000000 Bootstraps where b, c, d were rational numbers ranging from −10 to 10. The values were selected so the *apm2/a* ratio produced values between 0.999 and 1.001. The selected values were $b = 1.374$, $c = 1.6$, $d = 2.4$. Therefore, the equation (11) is:

$$apm2 = \ell\sqrt{0.02533n^2 - 0.125}\, (1 - (\frac{1}{1.374n^2 + 1.6n + 2.4})) \tag{11}$$

In what follows we will analyse the ratio between the apothem with estimated apothem, modeled apothem1 and modeled apothem 2 given by the ration between equations (7)/(1), (9)/(1), and (11)/(1), respectively:

Figure 3: A. Dot plot showing the ratio between the estimated apothem/apothem (*ape/a*) in the y-axis and the polygon sides (x-axis). B. Dot plot showing the ratio between the modeled apothem 1/apothem (*apm1/a*) (9)/(1). C. Dot plot showing the ratio between modeled apothem 2/apothem (*apm2/a*) (11)/(1). The points located between the dashed horizontal lines represent polygons predicted with accuracy equal to or higher than 0.99. The points represented in the interval between the dotted horizontal lines represent polygons predicted with accuracy equal to or greater than 0.999. Note the increased accuracy of the modeled apothems. The values were obtained for polygons of side length equal to 1. l = side, a = apothem, h = hypotenuse.



SOURCE: The authors

4 Testing the precision of the formula for corrected apothem estimation (*apm2*)

To test the precision of the corrected apothem estimation formula (*apm2*) we have calculated the area of a regular polygon with n = 100000 sides and side lengths values (l) varying between 0.0001 and 10000. The areas obtained with *apm2* (11) were compared with the areas obtained with a (1) (Table 1, columns 2 and 3).

Table 1 - Areas of regular polygons with 100000 sides obtained with the apothem (1) and corrected estimated apothem (11). In columns 2 and 3 the areas were estimated by the product of the perimeter and the apothem and in columns 4 and 5 the areas were estimated by the area of a circumference of radius equal to the apothem. Since the area of a polygon with 100000 sides approaches the area of a circumference, the areas were also estimated using the formula $\pi * a^2$. n = number of sides, l = side length, a = apothem.

l	$(n\ a\ l)/2$	$(n\ (apm2)\ l)/2$	$\pi\ a^2$	$\pi\ (apm2^2)$
0.0001	7.957747	7.957754	7.957747	7.957701
0.01	79577.47	79577.54	79577.47	79577.01
1	795774715	795775419	795774715	795770067
100	7.95774e+12	7.95775e+12	7.957747e+12	7.957701e+12
10000	7.95774e+16	7.95775e+16	7.957747e+16	7.957701e+16

Table 2 - Areas of regular polygons with $l = 1$ and variable side numbers obtained for the apothem (a, 1) and estimated apothem (*apm2*, 11). In columns 2 and 3 the areas were estimated by the product

of the perimeter and the apothem and in columns 4 and 5 the areas were estimated by the area of a circumference of radius equal to the apothem. n = number of sides, l = side length ,a = apothem.

To test the precision of the corrected apothem estimation formula *(apm2)*, we have calculated the areas of regular polygons with side length values of 1 and number of sides (n) varying between 3 to 729. The results are presented in Table 2.

n	*(n a l)/2*	*(n (apm2) l)/2*	$\pi\ a^2$	$\pi\ (apm2^2)$
3	0.4330127	0.4330365	0.2617994	0.2618281
9	6.181824	6.1783394	5.928682	5.931694
27	57.74994	57.75478	57.48909	57.49873
81	521.846	521.849	521.5843	521.5916
729	42290.47	42290.23	42290.21	42289.73

We have also compared the precision of apothem calculation between the corrected apothem *(apm2)* and the formula obtained by the Taylor polynomial series given by:

$$Taylor = \frac{n\ell}{2\pi}(1 - (\frac{(\pi)^2}{3n^2}) - \frac{(\pi)^4}{3n^4})) \tag{12}$$

Table 3 - Comparison between apothem obtained with the methods of *apm2* (11) and Taylor series polynomial approximation (12) in regular polygons with side length (l) = 1 and a number of sides (n) varying between 3 and 900.

n	*a*	*Taylor*	*apm2*
3	0.288667	0.290172	0.288691
9	1.373739	1.373744	1.374088
81	12.8850	12.8850	12.8851
900	143.2389	143.2389	143.238

5 Conclusion

Obtaining the apothem of regular polygons can be useful in academic tasks or practical situations. The apothem can be derived using equations that employ the tangent function or through equations obtained by polynomial series approximation. In the present work, we demonstrate an approach that deviates from these classic methods. Based on a nearly linear association between the number of sides (n) and the length of the side(l) we have developed a equation (11) that estimates the apothem of regular polygons. The equation was developed using a two-step approximation procedure. In the first step, we used linear regression to find the best fit line equation that allows an approximation of apothem using l and n. In the second step, we used a 2-round polynomial approximation bootstrap-based modeling with resampling of coefficients. This procedure produced a simple implementation algebraic equation that allows the estimation of the apothem of any regular polygon given only the length of one side (l) and the number of sides (n). This step allowed a minimum precision of 0.9987 for regular polygons of 5, 6 and 7 sides, and a precision higher than 0.999 for the other polygons.

The estimated apothem allied with the side length and number of sides allows the estimation of the area of any regular polygon without the use of trigonometric functions. The precision of equation (11) is similar to the approximation using Taylor polynomial series (12).

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